

Comparison of the Oscillatory Behaviors of a Gravitating Nambu-Goto String and a Test String

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Comparison of the oscillatory behavior of a gravitating infinite Nambu-Goto string and a test string is investigated using the general relativistic gauge invariant perturbation technique with two infinitesimal parameters on a flat spacetime background. Due to the existence of the pp-wave exact solution, we see that the conclusion that the dynamical degree of freedom of an infinite Nambu-Goto string is completely determined by that of gravitational waves, which was reached in our previous works [K. Nakamura, A. Ishibashi and H. Ishihara, Phys. Rev. D **62** (2002), 101502(R); K. Nakamura and H. Ishihara, Phys. Rev. D **63** (2001), 127501.], do not contradict to the dynamics of a test string. We also briefly discuss the implication of this result.

§1. Introduction

Nambu-Goto membranes are one of the simplest models of the extended objects that appear in various physical contexts. Particularly, the gravitational fields of extended objects provide interesting and important topics, including the dynamics of topological and/or non-topological defects (domain walls and cosmic strings^{3),4)}) and the gravitational waves emitted from them, the simplest type of brane worlds (vacuum brane⁵⁾).

It is well known that if the self-gravity of membranes is ignored, which is called “the test membrane case,” the equation of motion derived from the Nambu-Goto action admits oscillatory solutions, and membranes oscillate freely. However, recent investigations using exactly soluble models^{6),7),8)} have revealed that the oscillatory behavior of gravitating Nambu-Goto walls differs from that of test walls. In particular, gravitating walls cannot oscillate freely. The main points of these works are essentially separated into two parts: First, they showed that *the dynamical degree of freedom of the perturbative wall oscillations is completely determined by that of the gravitational waves*. Second, considering the gravitational wave scattering by a gravitating wall, they showed that *there is no resonance pole, except for damping modes*.

Such analysis was later extended to the case of the perturbative oscillations of an infinite Nambu-Goto string.^{1),2)} The results of this extension are summarized as follows. *The dynamical degree of freedom of the string oscillations is completely determined by that of gravitational waves* (as in the wall cases), and *an infinite string can oscillate continuously, but such oscillations simply represent the propagation of*

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gravitational waves along the string. The gravitational waves along the string are just the pp-wave exact solution to the Einstein equation, called cosmic string traveling waves. Note that the existence of the pp-wave solution is closely related to the specific symmetry of the spacetime. Therefore, it is natural to conjecture that the existence of the continuous oscillations of an infinite string is due to the special symmetry of the model and that the traveling wave solutions do not exist in more generic situations. Of course, this needs to be proven and it is merely speculation here.

In any case, from the results of perturbation analyses around exact solutions, it is reasonable to regard that the dynamical degree of freedom of string oscillations is that of gravitational waves. This implies that (A) *the oscillatory behavior of gravitating membranes might differ from those of test membranes even if their energy densities are very small*, because the background spacetimes of these models are exact solutions to the Einstein equation. However, if we regard the Nambu-Goto string as an idealization of GUT scale cosmic strings, their deficit angle is roughly approximated to be $\sim 10^{-6}$. Then, the following conjecture also seems natural. (B) *If the energy densities of membranes are very small, it should be possible to describe their gravitational fields as perturbations with a Minkowski spacetime background and describe their dynamics as those of test membranes.* However, the statements (A) and (B) seem to contradict each other.

The aim of this article is to compare the oscillatory behavior of a gravitating string and that of a test string. We concentrate on the dynamics of an infinite Nambu-Goto string, using general relativistic perturbation theory on a Minkowski background spacetime. We develop the gauge invariant perturbation theory with two infinitesimal parameters on the Minkowski spacetime background. We denote one of the perturbation parameters by ϵ , which corresponds to the string oscillation amplitude, and the other by λ , which corresponds to the string energy density. To determine the motion of the string, we solve the Einstein equation order by order. Physically, the perturbation in ϵ describes the oscillations of a string without its gravitational field. This corresponds to the oscillations of a test string. The perturbations in λ describes the gravitational field of an infinite static string, without oscillations. The oscillations of a gravitating string, the emission of gravitational waves due to these oscillations, and its radiation reaction are described by the simultaneous perturbations in ϵ and λ .

In this article, we give the perturbative analyses to $O(\epsilon)$, $O(\lambda)$, and $O(\epsilon\lambda)$ to compare with the analyses in our previous works.^{1),2)} The previous works are based on the first order perturbation with respect to the string oscillation amplitude. Therefore, $O(\epsilon\lambda)$ perturbations are the lowest order perturbations that are appropriate to compare with the oscillatory behavior of a gravitating string. We show that the $O(\epsilon\lambda)$ Einstein equations are almost same as those of the perturbative equations in our previous work. Further, we find that due to the existence of the above-mentioned pp-wave exact solution, the oscillations of a test infinite string are not contrary to the dynamics of a gravitating infinite string, at least to first order in the oscillation amplitude of the string.

The organization of this article is as follows. In the next section (§2), we briefly

comment on the energy-momentum tensor for the regularized Nambu-Goto string in order to study the dynamics of a gravitating Nambu-Goto string. To consider a gravitating string, we should consider a regularized string rather than an infinitesimally thin string. In §3, the gauge invariant perturbation technique with two infinitesimal parameters in general relativity is developed. In §4, we give the solutions to the $O(\epsilon)$ and $O(\lambda)$ Einstein equations. The $O(\epsilon\lambda)$ Einstein equations are given in §5. Comparison with the results of our previous works^{1),2)} is made in §6. The final section (§7) is devoted to a summary and discussion.

Throughout this paper, we denote Newton's gravitational constant by G and we use units such that the light velocity c is 1. Further, we use abstract index notation.⁹⁾

§2. Self-gravitating regularized Nambu-Goto string

To consider the dynamics of gravitating extended objects of co-dimension two (or larger), we must treat a delicate problem in general relativity. It is well known that there is no simple prescription of an arbitrary line source where a metric becomes singular.¹⁰⁾ Although it seems plausible that the string dynamics are well approximated by the dynamics of conical singularities, it has been shown that the world sheet of conical singularities must be totally geodesic.¹¹⁾ This is called Israel's Paradox. This implies that it is impossible for a generic Nambu-Goto string can be idealized in terms of conical singularities.

One of the simplest procedures to avoid this delicate problem is to introduce the string thickness.¹²⁾ As discussed in our previous papers,^{1),2)} we first consider a thick Nambu-Goto string, in which the singularity is regularized. Using this regularized string, we consider the perturbative dynamics of the thick string by solving the Einstein equations. Next, as in our previous works,^{1),2)} we consider the situation of a thin string, in which the string thickness is much smaller than the curvature scale of the bending string world sheet.

The energy-momentum tensor of a "thick string" is obtained as the extension of that for an infinitesimally thin string (see Appendix A.2). Let us consider a four-dimensional spacetime (\mathcal{M}, g_{ab}) including a "thick string." The energy-momentum tensor for an infinitesimally thin string has support only on its world sheet Σ_2 , which is a two-dimensional hypersurface in \mathcal{M} . When we consider the spacetime including an infinitesimally thin string, we assume that the region in the neighborhood of Σ_2 in the spacetime \mathcal{M} (at least in the neighborhood of Σ_2) can be foliated by two-dimensional surfaces. The tangent space of the spacetime \mathcal{M} in the neighborhood of Σ_2 can be decomposed into two two-dimensional subspaces, as seen in Appendix A.1. To introduce the thickness of the string, we consider a compact region \mathcal{D} in the complement space of Σ_2 , and we regard the "thick string world sheet" to be $\mathcal{D} \times \Sigma_2$ (see Appendix A.3).

In this article, we consider the energy-momentum tensor defined by

$$T_{ab} := -\rho q_{ab} \quad (2.1)$$

as that for the "thick string," where ρ is the string energy density and q_{ab} is an extension of the intrinsic metric on Σ_2 . The string energy density ρ and the intrinsic

metric q_{ab} are extended so that they have support on the “thick string world sheet” $\mathcal{D} \times \Sigma_2$, as seen in Appendix A.3. The metric γ_{ab} on the complement space of the “thick string” is defined by

$$\gamma_{ab} := g_{ab} - q_{ab}. \quad (2.2)$$

The rank of both the metrics q_{ab} and γ_{ab} is two. The string thickness is characterized by the support of the energy density ρ .

The divergence of this energy-momentum tensor, $\nabla_b T_a{}^b = 0$, is given by

$$q^{ab} \nabla_b \rho + \rho \gamma^{bc} \nabla_b q_c{}^a = 0, \quad (2.3)$$

$$\rho q^{bc} \nabla_b q_c{}^a =: \rho K^a = 0, \quad (2.4)$$

where K^a is the extrinsic curvature of the “thick string world sheet” defined in Appendix A. The Equation (2.3) is the continuity equation, which arises due to the introduction of the string thickness, and Eq. (2.4) is identical to with the equation of motion derived from the Nambu-Goto action. Hence, to study the dynamics of a gravitating Nambu-Goto string, we can concentrate only on the Einstein equation with the energy-momentum tensor (2.1). After solving the Einstein equation, we consider the thin string situation, if necessary.

§3. General relativistic two-parameter perturbation

Here, we develop the perturbation theory with two perturbation parameters in general relativity. In this article, we consider the perturbative oscillations of an infinite Nambu-Goto string using the Einstein equation with the above energy-momentum tensor (2.1). The background for the perturbation considered here is the Minkowski spacetime, and we have two infinitesimal parameters for the perturbation. One corresponds to the string oscillation amplitude, denoted by ϵ , and the other corresponds to the string energy density, denoted by λ .

As mentioned in the Introduction (§1), we give the analyses for $O(\epsilon)$, $O(\lambda)$, and $O(\epsilon\lambda)$ perturbations. As the second order perturbation in addition to $O(\epsilon\lambda)$, we may consider $O(\lambda^2)$ and $O(\epsilon^2)$ perturbations. $O(\lambda^2)$ perturbations describe the static gravitational field of an infinite string and $O(\epsilon^2)$ perturbations describe the test string oscillations of second order with respect to the string oscillation amplitude. Clearly, these perturbations have nothing to do with gravitational waves. Further, since our analyses are order by order, $O(\lambda^2)$ and $O(\epsilon^2)$ perturbations do not affect the $O(\epsilon\lambda)$ results. These are independent of each other. For these reasons, we do not consider $O(\lambda^2)$ and $O(\epsilon^2)$ perturbations here.

3.1. Background spacetime and its tangent space

Let us denote the background Minkowski spacetime by $(\mathcal{M}_0, \eta_{ab})$. When we consider the perturbative oscillations of the string, it is convenient to decompose the entire background spacetime into two submanifolds. One of the submanifolds contains the world sheet of an infinite straight string. We denote this submanifold by $(\mathcal{M}_1, \bar{q}_{ab})$. The other submanifold is the complement space of \mathcal{M}_1 . We denote this submanifold by $(\mathcal{M}_2, \bar{\gamma}_{ab})$. The entire background spacetime is given by the product of these manifolds, i.e., $\mathcal{M}_0 = \mathcal{M}_1 \times \mathcal{M}_2$.

The metric \bar{q}_{ab} on \mathcal{M}_1 is the intrinsic metric of the straight test string. The tensor $\bar{q}_a{}^b = \bar{q}_{ac}\eta^{cb}$ projects the tensor fields on the tangent space of the background \mathcal{M}_0 into those of \mathcal{M}_1 . We also introduce the indices i, j, \dots for the components of the tangent space of \mathcal{M}_1 , so that

$$\bar{q}_{ab} = \bar{q}_{ij}(d\sigma^i)_a(d\sigma^j)_b = -(dt)_a(dt)_b + (dz)_a(dz)_b. \quad (3.1)$$

The metric $\bar{\gamma}_{ab}$ on \mathcal{M}_2 is defined by $\bar{\gamma}_{ab} := \eta_{ab} - \bar{q}_{ab}$. We also introduce the indices α, β, \dots for the components of the tangent space of the complement space, which is normal to the background string world sheet, so that

$$\bar{\gamma}_{ab} = \bar{\gamma}_{\alpha\beta}(d\xi^\alpha)_a(d\xi^\beta)_b = (dx)_a(dx)_b + (dy)_a(dy)_b. \quad (3.2)$$

We also apply the Einstein convention for the summation over the indices i, j, \dots and α, β, \dots . This is different from the abstract index notation developed in the textbook by Wald.⁹⁾

Henceforth, we denote the metric \bar{q}_{ab} (\bar{q}_{ij}) on \mathcal{M}_1 (and its components) by q_{ab} (q_{ij}), for simplicity. Further, we denote the metrics $\bar{\gamma}_{ab}$ on \mathcal{M}_2 and its components by simply γ_{ab} and $\gamma_{\alpha\beta}$, respectively. We also denote the covariant derivative associated with the background metrics η_{ab} , q_{ab} and γ_{ab} by ∇_a , \mathcal{D}_a and D_a , respectively. Though this notation was originally introduced for use on the physical spacetime \mathcal{M} , we use it throughout this paper except in §2 and Appendix A.

3.2. Perturbative variables

Now, we introduce perturbative variables for the $O(\epsilon)$, $O(\lambda)$, and $O(\epsilon\lambda)$ perturbations.

The spacetime metric g_{ab} is expanded as

$$g_{ab} = \eta_{ab} + \epsilon h_{ab} + \lambda l_{ab} + \epsilon\lambda k_{ab}. \quad (3.3)$$

We can calculate the perturbative curvatures of $O(\epsilon)$, $O(\lambda)$ and $O(\epsilon\lambda)$ straightforwardly by starting from this expansion.

The physical meanings of the infinitesimal perturbation parameters ϵ and λ are clarified when we consider the perturbations of the energy-momentum tensor. We expand the energy-momentum tensor for an infinite Nambu-Goto string as follows:

$$T_a{}^b = -\lambda\rho q_a{}^b - \epsilon\lambda\left(\rho\delta q_a{}^b + \delta\rho q_a{}^b\right). \quad (3.4)$$

Since we consider the perturbation with Minkowski spacetime background, the energy density ρ of the string is also a perturbative variable. To make explicit that “ ρ is small”, we replace $\lambda\rho$ in the energy-momentum tensor (3.4) by ρ . Hence, λ is the infinitesimal perturbation parameter that corresponds to the energy density of the string.

The $O(\epsilon)$ perturbation $\delta q_a{}^b$ of the intrinsic metric $q_a{}^b$ in Eq. (3.4) is given by

$$\delta q_a{}^b = \gamma_{ac}q_d{}^b h^{cd} - \gamma_{\alpha\beta}\eta^{db}(\zeta^\alpha)_d q_a{}^c \nabla_c \xi_1^\beta - \gamma_{\alpha\beta}(\zeta^\alpha)_a q_c{}^b \nabla^c \xi_1^\beta, \quad (3.5)$$

where $(\zeta^\alpha)_a := \nabla_a \xi^\alpha = (d\xi^\alpha)_a$ is the coordinate basis introduced in Eq. (3.2). The

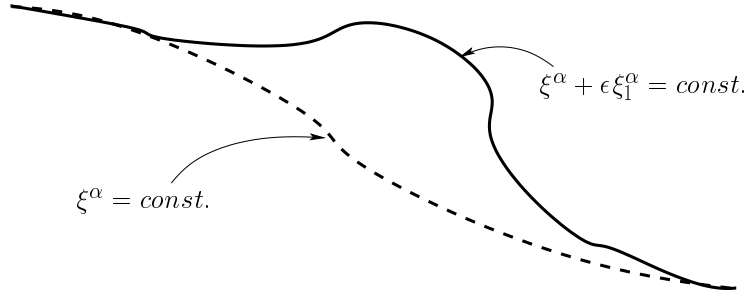


Fig. 1. Deformation of the string world sheet. The dashed curve is the background string world sheet $\xi^\alpha = \text{const.}$, and the solid curve is the perturbed string world sheet $\xi^\alpha + \epsilon \xi_1^\alpha = \text{const.}$

background string world sheet is the two-dimensional surface on which all coordinate functions ξ^α are constant. As depicted in Fig. 1, the perturbed string world sheet is also the two-dimensional surface on which $\xi^\alpha + \epsilon \xi_1^\alpha$ is constant. The second and third terms on the right-hand side of Eq. (3.5) are induced by the perturbative displacement ξ_1^α . Thus, the parameter ϵ is associated with the string oscillation amplitude. Since we regard the metric perturbation h_{ab} as $O(\epsilon)$, we should consider the perturbation h_{ab} to be induced by the perturbative displacement ξ_1^α .

Further, $\delta\rho$ in Eq. (3.4) should be regarded as $O(\epsilon\lambda)$. We should consider the continuity equation (2.3) as being due to the introduction of the string thickness. Equation (2.3) implies that the perturbative deformation of the string induces a perturbative energy density. Since the energy density ρ is the $O(\lambda)$ perturbative variable itself, the first-order perturbation of the energy density should be regarded as $O(\epsilon\lambda)$ or $O(\lambda^2)$ order. Since $O(\lambda^2)$ perturbations have nothing to do with the dynamics of the string, we regard $\delta\rho$ in Eq. (3.4) to be $O(\epsilon\lambda)$.

Starting from the perturbative expansions (3.3) and (3.4), we calculate the perturbative Einstein equations order by order. We also note that each order metric perturbation includes gauge freedom that is irrelevant to the physical perturbations. The perturbative variables $\delta\rho$ and ξ_1^α also include gauge freedom. To study the oscillations of the string and compare with the result in our previous work,^{1),2)} we must completely exclude such gauge freedom. To do this, we develop the perturbation theory with two infinitesimal parameters in a gauge invariant manner. To accomplish this, we first give that gauge transformation rules of the perturbative variables from general point of view. Then, we construct the gauge invariant variables for the perturbations defined by (3.3)–(3.5).

3.3. Gauge transformations

Let us consider the background spacetime $(\mathcal{M}_0, \eta_{ab})$ and a physical spacetime (\mathcal{M}, g_{ab}) , which we attempt to describe as a perturbation of the background spacetime. In relativistic perturbation theory, we are accustomed to the expressions of the forms (3.4) and (3.3) that are the relations between a tensor field (such as the metric) on the physical spacetime, the background value of the same field, and its

perturbation. Let us formally represent these expressions by

$$Q(x) = Q_0(x) + \delta Q(x), \quad (3.6)$$

symbolically. In this expression, we are implicitly assigning a correspondence between points of the perturbed and the background spacetimes. Moreover, the perturbed and unperturbed variables at the “same” point “ x ” are also defined implicitly. In the expression (3.6), we are implicitly considering the map $\mathcal{M}_0 \rightarrow \mathcal{M}$: $p \in \mathcal{M}_0 \mapsto q \in \mathcal{M}$. This correspondence associated with the map $\mathcal{M}_0 \rightarrow \mathcal{M}$ is what is usually called a gauge choice in the context of perturbation theory.¹³⁾ Clearly, this is more than the usual assignment of coordinate labels to points of the single spacetime. Furthermore, the correspondence established by relations such as (3.6) is not unique by itself, but, rather, (3.6) involves the degree of freedom of the choice of the map $\mathcal{M}_0 \rightarrow \mathcal{M}$ (i.e., the choice of the point identification map $\mathcal{M}_0 \rightarrow \mathcal{M}$). This is called gauge freedom. Further, this freedom always exists in the perturbation of a theory in which we impose general covariance.

Following this understanding of gauge freedom, Bruni et al¹⁴⁾ derived the gauge transformation rules up to fourth order. Here, we give their $O(\epsilon)$, $O(\lambda)$, and $O(\epsilon\lambda)$ results. At these orders, we must consider three kinds of gauge transformation rules for the perturbation of a physical variable Q on the physical manifold \mathcal{M} . We denote $O(\epsilon)$, $O(\lambda)$ and $O(\epsilon\lambda)$ order perturbations under the gauge $\mathcal{X} : \mathcal{M} \rightarrow \mathcal{M}_0$ by $\mathcal{X}\delta^{(\epsilon)}Q$, $\mathcal{X}\delta^{(\lambda)}Q$ and $\mathcal{X}\delta^{(\epsilon\lambda)}Q$, respectively. We consider the gauge transformation $\Phi := \mathcal{X}^{-1} \circ \mathcal{Y} : \mathcal{M}_0 \rightarrow \mathcal{M}_0$ (the gauge transformation from \mathcal{X} to \mathcal{Y}). Under this gauge transformation, the perturbative variables are transformed as follows:

$$\begin{aligned} \mathcal{Y}\delta^{(\epsilon)}Q - \mathcal{X}\delta^{(\epsilon)}Q &= \mathcal{L}_{(\epsilon)\xi}Q_0, \\ \mathcal{Y}\delta^{(\lambda)}Q - \mathcal{X}\delta^{(\lambda)}Q &= \mathcal{L}_{(\lambda)\xi}Q_0, \\ \mathcal{Y}\delta^{(\epsilon\lambda)}Q - \mathcal{X}\delta^{(\epsilon\lambda)}Q &= \mathcal{L}_{(\epsilon)\xi}\mathcal{X}\delta^{(\lambda)}Q + \mathcal{L}_{(\lambda)\xi}\mathcal{X}\delta^{(\epsilon)}Q \\ &\quad + \left\{ \mathcal{L}_{(\epsilon\lambda)\xi} + \frac{1}{2}\mathcal{L}_{(\epsilon)\xi}\mathcal{L}_{(\lambda)\xi} + \frac{1}{2}\mathcal{L}_{(\lambda)\xi}\mathcal{L}_{(\epsilon)\xi} \right\} Q_0, \end{aligned} \quad (3.7)$$

where $^{(\epsilon)}\xi$, $^{(\lambda)}\xi$ and $^{(\epsilon\lambda)}\xi$ are the $O(\epsilon)$, $O(\lambda)$ and $O(\epsilon\lambda)$ generators of the gauge transformation Φ , respectively. Gauge transformations of $O(\epsilon)$ and $O(\lambda)$ have well-known forms. Though the results derived by Bruni et al¹⁴⁾ include superfluous parameters at $O(\epsilon\lambda)$, it is easy to check that these parameters can be fixed through the replacement of the $O(\epsilon\lambda)$ generator $^{(\epsilon\lambda)}\xi$ of Φ , and doing so, the transformation rules (3.7) are obtained.¹⁵⁾ In terms of the passive coordinate transformation, the above gauge transformation Φ is given by

$$\begin{aligned} \tilde{x}^\mu &= x^\mu(q) - \lambda^{(\lambda)}\xi^\mu - \epsilon^{(\epsilon)}\xi^\mu \\ &\quad + \epsilon\lambda \left\{ -^{(\epsilon\lambda)}\xi^\mu + \frac{1}{2}^{(\lambda)}\xi^\nu \partial_\nu ^{(\epsilon)}\xi^\mu + \frac{1}{2}^{(\epsilon)}\xi^\nu \partial_\nu ^{(\lambda)}\xi^\mu \right\}. \end{aligned} \quad (3.8)$$

Inspecting the transformation rules (3.7), we develop a gauge invariant perturbation theory on Minkowski spacetime.

3.4. Gauge invariant variables

Inspecting the gauge transformation rules (3.7), we define the gauge invariant perturbative variables from the perturbed variables h_{ab} , l_{ab} , k_{ab} , ξ_1^a and $\delta\rho$. To carry this out, we use the procedure to construct the gauge invariant variables developed in a forthcoming paper.¹⁵⁾ The $O(\lambda)$ string energy density ρ is also a perturbative variable. Since the background value of the string energy density is trivial,¹³⁾ $O(\lambda)$ string energy density ρ is gauge invariant itself.

First, we consider the metric perturbations h_{ab} and l_{ab} of $O(\epsilon)$ and $O(\lambda)$, respectively. Under the gauge transformation $\Phi = \mathcal{X}^{-1} \circ \mathcal{Y}$, these metric perturbations are transformed as

$$\mathcal{Y}h_{ab} - \mathcal{X}h_{ab} = 2\nabla_{(a}^{(\epsilon)}\xi_{b)}, \quad (3.9)$$

$$\mathcal{Y}l_{ab} - \mathcal{X}l_{ab} = 2\nabla_{(a}^{(\lambda)}\xi_{b)}. \quad (3.10)$$

The metric perturbations h_{ab} and l_{ab} are decomposed as

$$h_{ab} =: \mathcal{H}_{ab} + 2\nabla_{(a}^{(\epsilon)}X_{b)}, \quad (3.11)$$

$$l_{ab} =: \mathcal{L}_{ab} + 2\nabla_{(a}^{(\lambda)}X_{b)}, \quad (3.12)$$

where the variables \mathcal{H}_{ab} , $^{(\epsilon)}X_a$, \mathcal{L}_{ab} and $^{(\lambda)}X_a$ are transformed as

$$\mathcal{Y}\mathcal{H}_{ab} - \mathcal{X}\mathcal{H}_{ab} = 0, \quad \mathcal{Y}^{(\epsilon)}X_a - \mathcal{X}^{(\epsilon)}X_a = ^{(\epsilon)}\xi_a, \quad (3.13)$$

$$\mathcal{Y}\mathcal{L}_{ab} - \mathcal{X}\mathcal{L}_{ab} = 0, \quad \mathcal{Y}^{(\lambda)}X_a - \mathcal{X}^{(\lambda)}X_a = ^{(\lambda)}\xi_a \quad (3.14)$$

under the gauge transformation $\Phi = \mathcal{X}^{-1} \circ \mathcal{Y}$. Clearly, \mathcal{H}_{ab} (\mathcal{L}_{ab}) is the gauge invariant part, and $^{(\epsilon)}X_b$ ($^{(\lambda)}X_b$) is the gauge variant part of the metric perturbation h_{ab} (l_{ab}). These decompositions are accomplished by carrying out the mode expansion, as shown in §4.2 and §5. The gauge invariant variables \mathcal{H}_{ab} and \mathcal{L}_{ab} have six independent components, while the original metric perturbations h_{ab} and l_{ab} have ten independent components.

Next, we consider the $O(\epsilon\lambda)$ metric perturbation k_{ab} , which transformed as

$$\begin{aligned} \mathcal{Y}k_{ab} - \mathcal{X}k_{ab} &= \mathcal{L}_{(\epsilon)\xi} \mathcal{X}l_{ab} + \mathcal{L}_{(\lambda)\xi} \mathcal{X}h_{ab} \\ &+ \left\{ \mathcal{L}_{(\epsilon\lambda)\xi} + \frac{1}{2}\mathcal{L}_{(\epsilon)\xi}\mathcal{L}_{(\lambda)\xi} + \frac{1}{2}\mathcal{L}_{(\lambda)\xi}\mathcal{L}_{(\epsilon)\xi} \right\} \eta_{ab} \end{aligned} \quad (3.15)$$

under the gauge transformation $\Phi = \mathcal{X}^{-1} \circ \mathcal{Y}$. Using the gauge variant parts $^{(\epsilon)}X_b$ and $^{(\lambda)}X_b$ of the $O(\epsilon)$ and $O(\lambda)$ metric perturbations, we first define the variable $\widehat{\mathcal{K}}_{ab}$ by

$$\begin{aligned} \widehat{\mathcal{K}}_{ab} &:= k_{ab} - \mathcal{L}_{(\epsilon)X} l_{ab} - \mathcal{L}_{(\lambda)X} h_{ab} \\ &+ \frac{1}{2} \{ \mathcal{L}_{(\epsilon)X} \mathcal{L}_{(\lambda)X} + \mathcal{L}_{(\lambda)X} \mathcal{L}_{(\epsilon)X} \} \eta_{ab}. \end{aligned} \quad (3.16)$$

We note that the tensor $\widehat{\mathcal{K}}_{ab}$ has ten independent components, like the usual metric. Further, under the gauge transformation $\Phi = \mathcal{X}^{-1} \circ \mathcal{Y}$, $\widehat{\mathcal{K}}_{ab}$ is transformed as

$$\mathcal{Y}\widehat{\mathcal{K}}_{ab} - \mathcal{X}\widehat{\mathcal{K}}_{ab} = \mathcal{L}_{\sigma}\eta_{ab}, \quad (3.17)$$

where

$$\sigma^a := {}^{(\epsilon\lambda)}\xi^a + \frac{1}{2} [{}^{(\lambda)}\xi, {}^{(\epsilon)}X]^a + \frac{1}{2} [{}^{(\epsilon)}\xi, {}^{(\lambda)}X]^a. \quad (3.18)$$

The gauge transformation (3.17) has exactly the same form as the linear-order metric perturbations. Then, using a procedure similar to that used to obtain the decompositions (3.11) and (3.12), we can decompose $\hat{\mathcal{K}}_{ab}$ as

$$\hat{\mathcal{K}}_{ab} =: \mathcal{K}_{ab} + 2\nabla_{(a} {}^{(\epsilon\lambda)}X_{b)}, \quad (3.19)$$

where the variables \mathcal{K}_{ab} and ${}^{(\epsilon\lambda)}X_a$ are transformed as

$$\mathcal{Y}\mathcal{K}_{ab} - \mathcal{X}\mathcal{K}_{ab} = 0, \quad \mathcal{Y}{}^{(\epsilon\lambda)}X_a - \mathcal{X}{}^{(\epsilon\lambda)}X_a = \sigma_a \quad (3.20)$$

under the gauge transformation $\Phi = \mathcal{X}^{-1} \circ \mathcal{Y}$. Thus, we can extract the gauge invariant part \mathcal{K}_{ab} from the $O(\epsilon\lambda)$ metric perturbation k_{ab} .

Finally, we define the gauge invariant variable for the displacement perturbation ξ_1^α and the energy density perturbation $\delta\rho$. The procedure to construct the corresponding gauge invariant variables for any perturbation is shown in the forthcoming paper.¹⁵⁾ The displacement perturbation ξ_1^α is the $O(\epsilon)$ perturbation of a scalar function ξ^α and is transformed as

$$\mathcal{Y}\xi_1^\alpha - \mathcal{X}\xi_1^\alpha = \mathcal{L}_{({}^{(\epsilon)}\xi)}\xi^\alpha = (\zeta^\alpha)_a {}^{(\epsilon)}\xi^a \quad (3.21)$$

under the gauge transformation $\Phi = \mathcal{X}^{-1} \circ \mathcal{Y}$. On the other hand, the energy density perturbation $\delta\rho$ is the $O(\epsilon\lambda)$ perturbation of the $O(\lambda)$ perturbation ρ . We note that the corresponding background value and the $O(\epsilon)$ perturbation are trivial. Then, under the gauge transformation $\Phi = \mathcal{X}^{-1} \circ \mathcal{Y}$, $\delta\rho$ is transformed as

$$\mathcal{Y}\delta\rho - \mathcal{X}\delta\rho = \mathcal{L}_{({}^{(\epsilon)}\xi)}\rho = {}^{(\epsilon)}\xi^a \nabla_a \rho. \quad (3.22)$$

Inspecting the gauge transformation rules (3.21) and (3.22), we define the variables

$$\hat{\Sigma} := \delta\rho - {}^{(\epsilon)}X^a \nabla_a \rho, \quad \hat{V}_a := \gamma_{\alpha\beta} (\zeta^\alpha)_a \xi_1^\beta - \gamma_{ab} {}^{(\epsilon)}X^b, \quad (3.23)$$

where $\hat{\Sigma}$ and \hat{V}_a are the gauge invariant variables corresponding to the energy density perturbation and the perturbative displacement of the string, respectively. Using these gauge invariant variables, we derive the perturbative Einstein equations of orders ϵ , λ and $\epsilon\lambda$.

§4. $O(\epsilon)$ and $O(\lambda)$ solutions

Here, we derive the $O(\epsilon)$ and $O(\lambda)$ solutions to the $O(\epsilon)$ and $O(\lambda)$ Einstein equations, respectively. To derive the perturbed Einstein equation for each order, the formulae for the curvature expansion given in Appendix B are useful. These can be derived by straightforward calculations.

4.1. $O(\epsilon)$ solutions

First, we consider the $O(\epsilon)$ solutions. Since there is no $O(\epsilon)$ term in the energy-momentum tensor (3.4), the $O(\epsilon)$ Einstein equations are the linearized vacuum equations. Explicitly, the $O(\epsilon)$ Einstein equations $\frac{\partial}{\partial \epsilon} G_a{}^b|_{\lambda=\epsilon=0} = 0$ are given by

$$\nabla_{[a} \mathcal{H}_{c]b}{}^c - \frac{1}{2} \eta_{ab} \nabla_{[c} \mathcal{H}_{d]}{}^{cd} = 0, \quad (4.1)$$

$$\mathcal{H}_{ab}{}^c := \nabla_{(a} \mathcal{H}_{b)}{}^c - \frac{1}{2} \nabla^c \mathcal{H}_{ab}. \quad (4.2)$$

At this order, the infinite string may oscillate but, it does not produce a gravitational field. The analysis for the $O(\epsilon)$ perturbations are completely parallel to the treatment of cylindrical and stationary perturbations at $O(\lambda)$ and that of the dynamical perturbations at $O(\epsilon\lambda)$. Through this analysis, we easily find that the gauge invariant metric perturbation \mathcal{H}_{ab} describes only the free propagation of gravitational waves, which has nothing to do with the string oscillations. We stipulate that there is no such gravitational wave. Hence, as the $O(\epsilon)$ solution, we obtain $\mathcal{H}_{ab} = 0$, and h_{ab} describes the pure gauge solution:

$$h_{ab} = 2\nabla_{(a}{}^{(\epsilon)} X_{b)}. \quad (4.3)$$

4.2. $O(\lambda)$ solutions

Next, we consider the $O(\lambda)$ solutions. Explicitly, the $O(\lambda)$ Einstein equations $\frac{\partial}{\partial \lambda} (G_a{}^b - 8\pi G T_a{}^b)|_{\lambda=\epsilon=0} = 0$ are given by

$$-2 \left(\nabla_{[a} \mathcal{L}_{c]b}{}^c - \frac{1}{2} \eta_{ab} \nabla_{[c} \mathcal{L}_{d]}{}^{cd} \right) = -8\pi G \rho q_{ab}, \quad (4.4)$$

$$\mathcal{L}_{ab}{}^c := \nabla_{(a} \mathcal{L}_{b)}{}^c - \frac{1}{2} \nabla^c \mathcal{L}_{ab}. \quad (4.5)$$

At this order, an infinite string is static, without oscillations, and it produces a static gravitational potential. In addition to this static potential, the linearized Einstein equations of $O(\lambda)$ also describe the free propagation of gravitational waves, as at $O(\epsilon)$. The derivation of this free propagation of gravitational waves is completely parallel to the analysis for the dynamical perturbations at $O(\epsilon\lambda)$. Here, we present only the treatment of cylindrical and stationary perturbations and the derivation of the static gravitational potential produced by the infinite string. The treatment given here also demonstrates the procedure to obtain the decomposition (3.11), (3.12), and (3.19) and thereby extract the gauge invariant part from the metric perturbations h_{ab} , l_{ab} and $\hat{\mathcal{K}}_{ab}$ in the case of cylindrical and stationary perturbations.

The cylindrical and stationary perturbations are characterized by

$$\mathcal{D}_a Q = 0, \quad (4.6)$$

where Q formally represents all perturbative variables of $O(\lambda)$. Though Eq. (4.4) is given in terms of gauge invariant variables, it is convenient to start from the $O(\lambda)$

metric perturbation l_{ab} itself. Since ∇_a is the covariant derivative associated with the flat metric, we easily check that

$$L_{ab}{}^c := \nabla_{(a} l_{b)}^c - \frac{1}{2} \nabla^c l_{ab} = \mathcal{L}_{ab}{}^c + \nabla_a \nabla_b^{(\lambda)} X^c. \quad (4.7)$$

Using the tensor $L_{ab}{}^c$, the right-hand side of Eq. (4.4), which is just $\frac{\partial}{\partial \lambda} G_a{}^b|_{\lambda=\epsilon=0}$, is given by

$$-2 \left(\nabla_{[a} \mathcal{L}_{c]b}{}^c - \frac{1}{2} \eta_{ab} \nabla_{[c} \mathcal{L}_{d]}^{cd} \right) = -2 \left(\nabla_{[a} L_{c]b}{}^c - \frac{1}{2} \eta_{ab} \nabla_{[c} L_{d]}^{cd} \right). \quad (4.8)$$

Under the gauge transformation $\Phi = \mathcal{X}^{-1} \circ \mathcal{Y}$, the $O(\lambda)$ metric perturbation l_{ab} is transformed as in Eq. (3.10). For cylindrical and stationary perturbations, the transformation rule (3.10) is given by

$$\mathcal{Y} l_{\alpha\beta} - \mathcal{X} l_{\alpha\beta} = D_\alpha^{(\lambda)} \xi_\beta + D_\beta^{(\lambda)} \xi_\alpha, \quad (4.9)$$

$$\mathcal{Y} l_{i\alpha} - \mathcal{X} l_{i\alpha} = D_\alpha^{(\lambda)} \xi_i, \quad (4.10)$$

$$\mathcal{Y} l_{ij} - \mathcal{X} l_{ij} = 0. \quad (4.11)$$

For cylindrical and stationary perturbations, the components of the $O(\lambda)$ Einstein equations (4.8) are given by

$$\begin{aligned} -D_\alpha D^\beta (l_i{}^i + l_\gamma{}^\gamma) + D^\gamma D_\alpha l_\gamma{}^\beta + D^\gamma D^\beta l_{\alpha\gamma} - \Delta l_\alpha{}^\beta \\ + \delta_\alpha{}^\beta \left(\Delta (l_\delta{}^\delta + l_i{}^i) - D^\gamma D^\delta l_{\delta\gamma} \right) = 0, \end{aligned} \quad (4.12)$$

$$D^\gamma D_\alpha l_\gamma{}^i - \Delta l_\alpha{}^i = 0, \quad (4.13)$$

$$-\Delta l_i{}^j + \delta_i{}^j \left(\Delta (l_\delta{}^\delta + l_l{}^l) - D^\gamma D^\delta l_{\gamma\delta} \right) = -16\pi G \delta_i{}^j \rho, \quad (4.14)$$

where $\delta_\alpha{}^\beta$ and $\delta_i{}^j$ are the two-dimensional Kronecker delta, and $\Delta := D^c D_c = D^\gamma D_\gamma$ is the Laplacian associated with the metric γ_{ab} .

From the trace of (4.12) and (4.14), we obtain

$$\Delta l_i{}^i = 0, \quad (4.15)$$

$$\Delta l_\delta{}^\delta - D^\gamma D^\delta l_{\gamma\delta} = -16\pi G \rho. \quad (4.16)$$

Further, substituting (4.15) and (4.16) into (4.14), we obtain

$$\Delta l_i{}^j = 0. \quad (4.17)$$

Here, we impose the condition that there is no singular behavior of $l_i{}^j$ on \mathcal{M}_2 . Then (4.17) implies that $l_i{}^j$ should be constant on \mathcal{M}_2 . This constant can be removed through a scale transformation of the coordinates σ^i . Then we may choose $l_i{}^j = 0$ without loss of generality.

Next, we consider Eq. (4.13). Here, we decompose $l_{\alpha i}$ as

$$l_{\alpha i} = D_\alpha f_i + \mathcal{E}_{\alpha\beta} D^\beta g_i, \quad (4.18)$$

where $\mathcal{E}_{\alpha\beta}$ is the component of the two-dimensional totally antisymmetric tensor defined by

$$\mathcal{E}_{ab} := \mathcal{E}_{\alpha\beta}(d\xi^\alpha)_a(d\xi^\alpha)_b := (dx)_a(dy)_b - (dy)_a(dx)_b. \quad (4.19)$$

From (4.10), it is clear that f_i represents the gauge freedom. Substituting the decomposition (4.18) into Eq. (4.13), we obtain

$$\mathcal{E}_{\alpha\beta}D^\beta\Delta g_i = 0. \quad (4.20)$$

This shows that Δg_i is a constant on \mathcal{M}_2 . If this constant is non-vanishing, then $l_{\alpha i}$ proportional to the distance from the axis of the cylindrical symmetry and diverges at infinity (i.e. in the region infinitely far from the string). Here, we impose the condition that there is no divergence of the perturbative metric. In this case, g_i is a solution to the two-dimensional Laplace equation on the flat space. We can easily check that if g_i is a solution to the two-dimensional Laplace equation, there is a function \tilde{f}_i such that

$$\mathcal{E}_{\alpha\beta}D^\beta g_i = D_\alpha \tilde{f}_i. \quad (4.21)$$

Therefore, g_i is also regarded as representing the gauge freedom. Thus, we obtain the solution to the equation (4.13)

$$l_{\alpha i} = D_\alpha f_i, \quad (4.22)$$

which represents only the gauge freedom of perturbations.

Finally, we consider the components $l_{\alpha\beta}$. We decompose $l_{\alpha\beta}$ as

$$l_{\alpha\beta} = \frac{1}{2}\gamma_{\alpha\beta}\Xi + D_\alpha D_\beta F + \mathcal{A}_{\alpha\beta}G, \quad (4.23)$$

where the derivative operator $\mathcal{A}_{\alpha\beta}$ is defined by

$$\mathcal{A}_{\alpha\beta} := \mathcal{E}_{\gamma(\alpha}D_{\beta)}D^\gamma. \quad (4.24)$$

We also decompose the generator $^{(\lambda)}\xi^a$ of the gauge transformation as

$$^{(\lambda)}\xi_\alpha = D_\alpha f + \mathcal{E}_{\alpha\beta}D^\beta g. \quad (4.25)$$

Using this decomposition, the gauge transformation (4.9) is given by

$$\mathcal{Y}l_{\alpha\beta} - \mathcal{X}l_{\alpha\beta} = D_\alpha D_\beta f + 2\mathcal{A}_{\alpha\beta}g. \quad (4.26)$$

This shows that the functions F and G in the decomposition (4.23) represent the gauge freedom, and only the function Ξ is gauge invariant. By substitution of the decomposition (4.23) and $l_i{}^j = 0$ into (4.12), we can easily check that (4.12) is trivial. Further, Eq. (4.16) with the decomposition (4.23) gives

$$\Delta\Xi = -32\pi G\rho. \quad (4.27)$$

Since Δ is the two-dimensional Laplacian, for a given function ρ , the solution Ξ to Eq. (4.27) can be obtained by using usual Green function of the differential

operator Δ on \mathcal{M}_2 . Further, we can easily see that the solution to Eq. (4.27) has a logarithmic divergence at infinity. However, this logarithmic divergence can be interpreted as the deficit angle produced by the string.³⁾ For this reason, we do not treat this divergence, seriously, and here we regard this logarithmic divergence as representing the asymptotically conical structure of the full spacetime (\mathcal{M}, g_{ab}) . Thus, using the solution to Eq. (4.27), the gauge invariant metric perturbation \mathcal{L}_{ab} is given by

$$\mathcal{L}_{ab} = \frac{1}{2}\gamma_{ab}\Xi. \quad (4.28)$$

Including the gauge freedom, we obtain the $O(\lambda)$ metric l_{ab} as the solution to the Einstein equations of $O(\lambda)$

$$l_{ab} = \frac{1}{2}\gamma_{ab}\Xi + 2\nabla_{(a}^{(\lambda)} X_{b)}, \quad (4.29)$$

at least for cylindrical and stationary perturbations. With analysis completely parallel to that used for the dynamical perturbations of $O(\epsilon\lambda)$, we can see that (4.29) is the general solution to the $O(\lambda)$ Einstein equations (4.8) if we stipulate that there is no free propagation of gravitational waves, which have nothing to do with the string oscillations.

When $\rho = 0$, we note that Ξ must be constant on \mathcal{M}_2 if we impose the metric perturbation l_{ab} has no divergence in \mathcal{M}_2 , except for the logarithmic divergence at infinity. This constant can be eliminated through a the conformal scale transformation of the coordinates ξ^α , and we may choose $\Xi = 0$ without loss of generality. This implies that $O(\epsilon)$ perturbations do not include cylindrical and stationary solutions, except for that representing the gauge freedom. Further, this also holds in $O(\epsilon\lambda)$ if $\delta\rho$ in (3.4) does not include any stationary components.

§5. $O(\epsilon\lambda)$ Einstein equations

In this section, we consider dynamical perturbations of $O(\epsilon\lambda)$. We derive the $O(\epsilon\lambda)$ Einstein equations using the mode expansions in the harmonics defined in Appendix C. The treatment of the dynamical perturbation in this section also shows that the procedure for the decomposition (3.11), (3.12) and (3.19) and the resulting extraction of the gauge invariant part from the metric perturbations h_{ab} , l_{ab} and $\hat{\mathcal{K}}_{ab}$ in the case of dynamical perturbations.

As mentioned in §4.1, we have imposed the condition that the free propagation of gravitational waves that have nothing to do with the string oscillations ; i.e., $\mathcal{H}_{ab} = 0$. Then, using the formula (B.5), we can derive the $O(\epsilon\lambda)$ Einstein equation $\frac{\partial^2}{\partial\lambda\partial\epsilon}(G_a{}^b - 8\pi GT_a{}^b)\Big|_{\lambda=\epsilon=0} = 0$ in terms of the gauge invariant variables defined by Eqs. (3.19) and (3.23):

$$\mathcal{K}_{ab}{}^c := \nabla_{(a}\mathcal{K}_{b)}{}^c - \frac{1}{2}\nabla^c\mathcal{K}_{ab}, \quad (5.1)$$

$$-2\left(\nabla_{[a}\mathcal{K}_{c]b}{}^c - \frac{1}{2}\eta_{ab}\nabla_{[c}\mathcal{K}_{d]}{}^{cd}\right) = 8\pi G\left(-q_{ab}\hat{\Sigma} + 2\rho q_{c(a}\nabla^c\hat{V}_{b)}\right). \quad (5.2)$$

To evaluate the Einstein equations (5.2), we must carry out the decomposition (3.19) by inspecting the gauge transformation rules (3.17). This is given by

$$\mathcal{Y}\widehat{\mathcal{K}}_{\alpha\beta} - \mathcal{X}\widehat{\mathcal{K}}_{\alpha\beta} = D_\alpha\zeta_\beta + D_\beta\zeta_\alpha, \quad (5.3)$$

$$\mathcal{Y}\widehat{\mathcal{K}}_{\alpha i} - \mathcal{X}\widehat{\mathcal{K}}_{\alpha i} = D_\alpha\zeta_i + \mathcal{D}_i\zeta_\alpha, \quad (5.4)$$

$$\mathcal{Y}\widehat{\mathcal{K}}_{ij} - \mathcal{X}\widehat{\mathcal{K}}_{ij} = \mathcal{D}_i\zeta_j + \mathcal{D}_j\zeta_i. \quad (5.5)$$

To accomplish the decomposition (3.19) for dynamical perturbations, it is convenient to use the expansion in harmonic functions summarized in Appendix C. The procedure to carry out the decompositions (3.11) and (3.12) is completely the same. As mentioned in Appendix C, we should distinguish perturbative modes for which κ defined by (C.3) vanishes from those for which it does not. The $\kappa = 0$ modes correspond to the perturbative modes that propagate along the string, and the $\kappa \neq 0$ modes are the other dynamical modes. Though the cylindrical and stationary perturbative modes also exist in the present case, as in $O(\lambda)$ perturbations, we do not treat these modes here. Clearly, these modes have nothing to do with the dynamics of the string. This can be easily seen from treatments that are completely the same as that in §4.2.

5.1. $\kappa \neq 0$ mode perturbation

First, we consider the case in which the all perturbative variables satisfy the conditions

$$\mathcal{D}_a Q \neq 0, \quad \mathcal{D}^a \mathcal{D}_a Q \neq 0, \quad (5.6)$$

where Q formally represents all the perturbative variables, $\widehat{\mathcal{K}}_{ab}$, $\widehat{\Sigma}$, \widehat{V}^a and ζ^a . Using the $\kappa \neq 0$ mode harmonics introduced in Appendix C.1, these perturbative variables are expanded as follows:

$$\widehat{\mathcal{K}}_{\alpha\beta} =: \int f_{\alpha\beta} S, \quad \widehat{\mathcal{K}}_{\alpha i} =: \int [f_{\alpha(o1)} V_{(o1)i} + f_{\alpha(e1)} V_{(e1)i}], \quad (5.7)$$

$$\widehat{\mathcal{K}}_{ij} =: \int [f_{(o2)} T_{(o2)ij} + f_{(e0)} T_{(e0)ij} + f_{(e2)} T_{(e2)ij}], \quad (5.8)$$

$$\zeta_\alpha =: \int \iota_\alpha S, \quad \zeta_i =: \int [\iota_{(o)} V_{(o)i} + \iota_{(e)} V_{(e)i}], \quad (5.9)$$

$$\widehat{\Sigma} =: -\frac{1}{32\pi G} \int \Sigma S, \quad \rho \widehat{V}_\alpha =: \frac{1}{16\pi G} \int V_\alpha S, \quad \rho \widehat{V}_i = 0, \quad (5.10)$$

where the measure \int of the harmonic analyses is given by $\int = \int d^2 k$. Throughout this subsection, we use the measure \int with this meaning. All expansion coefficients defined by (5.7)–(5.10) are tensors on $(\mathcal{M}_2, \gamma_{ab})$.

By inspecting the gauge transformation rules of the coefficients, we easily find gauge invariant combinations. In terms of the mode coefficients defined by (5.7)–(5.9) the gauge transformation rules (5.3)–(5.5) are given by

$$\mathcal{Y}f_{\alpha\beta} - \mathcal{X}f_{\alpha\beta} = D_\alpha \iota_\beta + D_\beta \iota_\alpha, \quad (5.11)$$

$$\mathcal{Y}f_{\alpha(e1)} - \mathcal{X}f_{\alpha(e1)} = D_\alpha \iota_{(e)} + \iota_\alpha, \quad (5.12)$$

$$\mathcal{Y} f_{(e0)} - \mathcal{X} f_{(e0)} = 2\kappa^2 \iota_{(e)}, \quad (5.13)$$

$$\mathcal{Y} f_{(e2)} - \mathcal{X} f_{(e2)} = 2\iota_{(e)}, \quad (5.14)$$

$$\mathcal{Y} f_{(o1)}^\alpha - \mathcal{X} f_{(o1)}^\alpha = D^\alpha \iota_{(o)}, \quad (5.15)$$

$$\mathcal{Y} f_{(o2)} - \mathcal{X} f_{(o2)} = 2\iota_{(o)}. \quad (5.16)$$

From these transformation rules, we see that there exist six independent gauge-invariant quantities. First, from Eqs. (5.15), (5.16) and (5.14), we see that there exist two independent gauge invariant quantities defined by

$$F_\alpha := f_{\alpha(o1)} - \frac{1}{2} D_\alpha f_{(o2)}. \quad (5.17)$$

Second, we define

$$F := f_{(e0)} - \kappa^2 f_{(e2)}. \quad (5.18)$$

The gauge transformations (5.13) reveal that F is a gauge invariant quantity. To find the remaining gauge invariants, we define the variable

$$Z^\alpha := f_{(e1)}^\alpha - \frac{1}{2} D^\alpha f_{(e2)}. \quad (5.19)$$

From Eqs. (5.12) and (5.14), Z^α is transformed as

$$\mathcal{Y} Z^\alpha - \mathcal{X} Z^\alpha = \iota^\alpha. \quad (5.20)$$

Hence, we can easily obtain the gauge-invariant combinations

$$F_{\alpha\beta} := f_{\alpha\beta} - 2D_{(\alpha} Z_{\beta)}. \quad (5.21)$$

From the gauge transformation rules (5.11) and (5.20) and this definition, we easily check that $F_{\alpha\beta}$ is gauge invariant.

Thus, we obtain the six gauge invariant variables F , F_α and $F_{\alpha\beta}$. We note that the other four components are gauge variant. In terms of these variables, the metric perturbations $\hat{\mathcal{K}}_{ab}$ are given by

$$\hat{\mathcal{K}}_{\alpha\beta} = \int (F_{\alpha\beta} + 2D_{(\alpha} Z_{\beta)}) S, \quad (5.22)$$

$$\hat{\mathcal{K}}_{\alpha i} := \int \left[\left(F_\alpha + \frac{1}{2} D_\alpha f_{(o2)} \right) V_{(o1)i} + \left(Z_\alpha + \frac{1}{2} D_\alpha f_{(e2)} \right) V_{(e1)i} \right], \quad (5.23)$$

$$\hat{\mathcal{K}}_{ij} := \int [f_{(o2)} T_{(o2)ij} + (F + \kappa^2 f_{(e2)}) T_{(e0)ij} + f_{(e2)} T_{(e2)ij}]. \quad (5.24)$$

This implies that we have accomplished the decomposition of $\hat{\mathcal{K}}_{ab}$ into \mathcal{K}_{ab} and $2\nabla_{(a}^{(\epsilon\lambda)} X_{b)}$ in the case of $\kappa \neq 0$ modes, obtaining

$$\mathcal{K}_{\alpha\beta} := \int F_{\alpha\beta} S, \quad \mathcal{K}_{\alpha i} := \int F_\alpha V_{(o1)i}, \quad \mathcal{K}_{ij} := \int F T_{(e0)ij} \quad (5.25)$$

and

$$^{(\epsilon\lambda)}X_\alpha := \int Z_\alpha S, \quad ^{(\epsilon\lambda)}X_i := \frac{1}{2} \int [f_{(o2)}V_{(o1)i} + f_{(e2)}V_{(e1)i}]. \quad (5.26)$$

Substituting Eqs. (5.10) and (5.25) into (5.2), all components of the $O(\epsilon\lambda)$ Einstein equations for $\kappa \neq 0$ modes are given by

$$(\Delta + \kappa^2)F = 0, \quad (5.27)$$

$$(\Delta + \kappa^2)F_{\alpha\beta} = 2 \left(D_{(\alpha}V_{\beta)} - \frac{1}{2}\gamma_{\alpha\beta}D_\gamma V^\gamma \right), \quad (5.28)$$

$$D^\gamma F_{\alpha\gamma} - \frac{1}{2}D_\alpha F = V_\alpha, \quad (5.29)$$

$$F_\gamma{}^\gamma = 0, \quad (5.30)$$

$$D^\gamma V_\gamma + \frac{1}{2}\Sigma = 0, \quad (5.31)$$

$$(\Delta + \kappa^2)F_\alpha = 0, \quad (5.32)$$

$$D_\alpha F^\alpha = 0. \quad (5.33)$$

These equations are almost the same as the linearized Einstein equations for $\kappa \neq 0$ modes obtained in our previous work.¹⁾ The differences are that there is no background curvature term in Eq. (5.28) and that the covariant derivative D_a is associated with the flat metric γ_{ab} , while D_a in our previous work is not.

5.2. $\kappa = 0$ mode perturbation

Here, we consider the case in which the metric perturbation $\hat{\mathcal{K}}_{ab}$, the energy density perturbation $\hat{\Sigma}$, the displacement of the string world sheet \hat{V}^a , and the generator ζ^a of the gauge transformations satisfy the equation

$$\mathcal{D}^a \mathcal{D}_a Q = 0, \quad \mathcal{D}_a Q \neq 0, \quad (5.34)$$

where Q formally represents each of these perturbative variables. In this case, using the $\kappa = 0$ modes harmonics introduced in Appendix C.2, these perturbative variables are expanded as

$$\hat{\mathcal{K}}_{\alpha\beta} =: \int f_{\alpha\beta} S, \quad \hat{\mathcal{K}}_{\alpha i} =: \int [f_{\alpha(l1)}V_{(l1)i} + f_{\alpha(e1)}V_{(e1)i}], \quad (5.35)$$

$$\hat{\mathcal{K}}_{ij} =: \int [f_{(l2)}T_{(l2)ij} + f_{(e0)}T_{(e0)ij} + f_{(e2)}T_{(e2)ij}], \quad (5.36)$$

$$\zeta_\alpha =: \int \iota_\alpha S, \quad \zeta_i =: \int [\iota_{(l)}V_{(l1)i} + \iota_{(e)}V_{(e1)i}], \quad (5.37)$$

$$\hat{\Sigma} =: -\frac{1}{32\pi G} \int \Sigma S, \quad \rho \hat{V}_\alpha =: \frac{1}{16\pi G} \int V_\alpha S, \quad \rho \hat{V}_i = 0, \quad (5.38)$$

where the measure \int of the Fourier integration is given by $\int = \sum_{\nu=\pm 1} \int d\omega$. Throughout this subsection, we denote this measure by \int . All expansion coefficients defined by (5.35)–(5.37) are tensors on $(\mathcal{M}_2, \gamma_{ab})$.

As in the case of $\kappa \neq 0$ mode perturbations, we can easily find gauge invariant combinations for the metric perturbations by inspecting the gauge transformation rules (5.3)–(5.5) for the coefficients. In terms of the mode coefficients defined by (5.35)–(5.37), the gauge transformation rules (5.3)–(5.5) are given by

$$\mathcal{Y} f_{\alpha\beta} - \mathcal{X} f_{\alpha\beta} = D_\alpha \iota_\beta + D_\beta \iota_\alpha, \quad (5.39)$$

$$\mathcal{Y} f_{\alpha(e1)} - \mathcal{X} f_{\alpha(e1)} = D_\alpha \iota_{(e)} + \iota_\alpha, \quad (5.40)$$

$$\mathcal{Y} f_{\alpha(l1)} - \mathcal{X} f_{\alpha(l1)} = D_\alpha \iota_{(l)}, \quad (5.41)$$

$$\mathcal{Y} f_{(l2)} - \mathcal{X} f_{(l2)} = 0, \quad (5.42)$$

$$\mathcal{Y} f_{(e0)} - \mathcal{X} f_{(e0)} = 4\omega^2 \iota_{(l)}, \quad (5.43)$$

$$\mathcal{Y} f_{(e2)} - \mathcal{X} f_{(e2)} = 2\iota_{(e)}. \quad (5.44)$$

From these transformation rules, we see that there exist six independent gauge-invariant quantities. First, Eq. (5.42) shows that $f_{(l2)}$ itself is a gauge invariant quantity:

$$H := f_{(l2)}. \quad (5.45)$$

Second, we define

$$H_\alpha := f_{\alpha(l1)} - \frac{1}{4\omega^2} D_\alpha f_{(e0)}. \quad (5.46)$$

The gauge transformation rules (5.41) and (5.43) imply that H_α is gauge invariant. To find the remaining gauge invariants, we define the variable

$$Z^\alpha := f_{(e1)}^\alpha - \frac{1}{2} D^\alpha f_{(e2)}. \quad (5.47)$$

This definition, along with the gauge transformation rules (5.40) and (5.44), shows that the variable Z^α transforms as

$$\mathcal{Y} Z^\alpha - \mathcal{X} Z^\alpha = \iota^\alpha. \quad (5.48)$$

Using the variable Z^α , we define

$$H_{\alpha\beta} := f_{\alpha\beta} - 2D_{(\alpha} Z_{\beta)}. \quad (5.49)$$

From this definition and the gauge transformation rules (5.39) and (5.48), we easily check that $H_{\alpha\beta}$ is gauge invariant.

From the above, we obtain the six gauge invariant variables H , H_α and $H_{\alpha\beta}$. We note that the other four components are gauge variant. Using these variables, the metric perturbations $\hat{\mathcal{K}}_{ab}$ are given by

$$\hat{\mathcal{K}}_{\alpha\beta} = \int (H_{\alpha\beta} + 2D_{(\alpha} Z_{\beta)}) S, \quad (5.50)$$

$$\hat{\mathcal{K}}_{\alpha i} = \int \left[\left(H_\alpha + \frac{1}{4\omega^2} D_\alpha f_{(e0)} \right) V_{(l1)i} + \left(Z_\alpha + \frac{1}{2} D_\alpha f_{(e2)} \right) V_{(e1)i} \right], \quad (5.51)$$

$$\hat{\mathcal{K}}_{ij} = \int [HT_{(l2)ij} + f_{(e0)} T_{(e0)ij} + f_{(e2)} T_{(e2)ij}]. \quad (5.52)$$

This implies that we have accomplished the decomposition of $\widehat{\mathcal{K}}_{ab}$ into \mathcal{K}_{ab} and $2\nabla_{(a}^{(\epsilon\lambda)} X_{b)}$ in the case of the $\kappa = 0$ modes, obtaining

$$\mathcal{K}_{\alpha\beta} := \int H_{\alpha\beta} S, \quad \mathcal{K}_{\alpha i} := \int H_{\alpha} V_{(l1)i}, \quad \mathcal{K}_{ij} := \int HT_{(l2)ij}, \quad (5.53)$$

and

$$^{(\epsilon\lambda)}X_{\alpha} := \int Z_{\alpha} S, \quad ^{(\epsilon\lambda)}X_i := \frac{1}{2} \int \left[\frac{1}{2\omega^2} f_{(e0)} V_{(l1)i} + f_{(e2)} V_{(e1)i} \right]. \quad (5.54)$$

Substituting Eqs. (5.38) and (5.53) into (5.2), all components of the $O(\epsilon\lambda)$ Einstein equations for the $\kappa = 0$ modes are obtained as

$$\Delta H_{\alpha\beta} = 2 \left(D_{(\alpha} V_{\beta)} - \frac{1}{2} \gamma_{\alpha\beta} D_{\gamma} V^{\gamma} \right), \quad (5.55)$$

$$D^{\gamma} H_{\gamma} + 2\omega^2 H = 0, \quad (5.56)$$

$$D^{\gamma} H_{\gamma}^{\alpha} + 2\omega^2 H^{\alpha} = V^{\alpha}, \quad (5.57)$$

$$\Delta H^{\alpha} = 0, \quad (5.58)$$

$$H_{\gamma}^{\gamma} = 0, \quad (5.59)$$

$$\Delta H = 0, \quad (5.60)$$

$$D_{\alpha} V^{\alpha} + \frac{1}{2} \Sigma = 0. \quad (5.61)$$

These equations are almost the same as the linearized Einstein equations for the $\kappa = 0$ modes in our previous work,²⁾ as in the case of the $\kappa \neq 0$ modes. The differences in this case are the same as those in the case of $\kappa \neq 0$ modes, i.e., those regarding the absence of the curvature terms and the meaning of the covariant derivative D_{α} .

On the basis of the Einstein equations (5.27)–(5.33) for $\kappa \neq 0$ modes and equations (5.55)–(5.61), we compare the oscillatory behavior of a gravitating string and a test string.

§6. Comparison with curved spacetime background analyses

Here, we compare the results of the present work, the Einstein equations (5.27)–(5.33) and (5.55)–(5.61) derived in the previous sections (§§5.1 and 5.2), with the results of our previous works.^{1),2)} Since the analyses to derive these equations are based on the harmonic expansion, as in the last section, we again treat the dynamical modes that propagate along the string (the $\kappa = 0$ modes) and others (the $\kappa \neq 0$ modes), separately. Of course, as the remaining perturbative modes, the cylindrical and stationary perturbative modes exist as seen in §4.2. However, these stationary perturbations have nothing to do with the dynamics of the Nambu-Goto membrane. For this reason we discuss only the $\kappa \neq 0$ and $\kappa = 0$ modes in this section.

6.1. $\kappa \neq 0$ mode solutions

For $\kappa \neq 0$ mode perturbations, the $O(\epsilon\lambda)$ Einstein equations are given by (5.27)–(5.33). These equations are almost same as the linearized Einstein equations for the

$\kappa \neq 0$ modes in Ref. 1), as we previously noted. The displacement perturbation V^α in (5.28) and (5.29) seems induce nontrivial perturbations of the metric. However, unlike in our previous work for $\kappa \neq 0$ mode perturbations, these equations lead to $V^\alpha = 0$ as we show in this section. Hence, we conclude that the $\kappa \neq 0$ mode metric perturbations have nothing to do with the string dynamics at $O(\epsilon\lambda)$. To show this, we first derive the solutions to the equations (5.27)–(5.33).

First, we consider Eqs. (5.32) and (5.33) for the variable F_α . We consider the decomposition of the vector on \mathcal{M}_2 as

$$F_\alpha = D_\alpha \Psi_{(o)} + \mathcal{E}_{\alpha\beta} D^\beta \bar{\Phi}_{(o)}. \quad (6.1)$$

Using (5.33), we obtain

$$D^\alpha F_\alpha = \Delta \Psi_{(o)} = 0. \quad (6.2)$$

This shows that $\Psi_{(o)}$ is a solution to the two-dimensional Laplace equation. Then, we can easily check that there exists a function \mathcal{I} such that

$$\epsilon_{\alpha\beta} D^\beta \mathcal{I} = D_\alpha \Psi_{(o)}, \quad (6.3)$$

and we obtain

$$F_\alpha = \epsilon_{\alpha\beta} D^\beta (\bar{\Phi}_{(o)} + \mathcal{I}) =: \mathcal{E}_{\alpha\beta} D^\beta \hat{\Phi}_{(o)}. \quad (6.4)$$

Substituting this into (5.32), we obtain

$$\begin{aligned} 0 &= (\Delta + \kappa^2) \mathcal{E}_{\alpha\beta} D^\beta (\bar{\Phi}_{(o)} + \mathcal{I}) \\ &= \mathcal{E}_{\alpha\beta} D^\beta (\Delta + \kappa^2) (\bar{\Phi}_{(o)} + \mathcal{I}). \end{aligned} \quad (6.5)$$

Integrating this equation, we obtain

$$(\Delta + \kappa^2) (\bar{\Phi}_{(o)} + \mathcal{I}) = C, \quad (6.6)$$

where C is a constant. This equation can also be written by

$$(\Delta + \kappa^2) \left(\bar{\Phi}_{(o)} + \mathcal{I} - \frac{1}{\kappa^2} C \right) = 0. \quad (6.7)$$

Here, we redefine the variable $\bar{\Phi}_{(o)}$ as

$$\Phi_{(o)} := \bar{\Phi}_{(o)} + \mathcal{I} - \frac{1}{\kappa^2} C. \quad (6.8)$$

Then, Eqs. (5.32) and (5.33) are reduced to

$$F_\alpha = \mathcal{E}_{\alpha\beta} D^\beta \Phi_{(o)}, \quad (\Delta + \kappa^2) \Phi_{(o)} = 0. \quad (6.9)$$

The scalar variable $\Phi_{(o)}$ represents the gravitational wave that corresponds to the odd mode gravitational wave in Ref. 1). These gravitational waves propagate freely and have nothing to do with the motion of the string.

Next, we consider Eqs. (5.27)–(5.31). First, (5.30) shows that the tensor $F_{\alpha\beta}$ is traceless. Therefore we can decompose $F_{\alpha\beta}$ as

$$F_{\alpha\beta} = \left(D_\alpha D_\beta - \frac{1}{2} \gamma_{\alpha\beta} \Delta \right) \bar{\Phi}_{(e)} + \mathcal{A}_{\alpha\beta} \Psi_{(e)}. \quad (6.10)$$

Substituting this into (5.29), we obtain

$$V_\alpha = \frac{1}{2}D_\alpha \Delta \bar{\Phi}_{(e)} + \frac{1}{2}\mathcal{E}_{\alpha\beta} D^\beta \Delta \Psi_{(e)} - \frac{1}{2}D_\alpha F. \quad (6.11)$$

Further, Eqs. (5.28) and (6.11) yield

$$0 = \left(D_\alpha D_\beta - \frac{1}{2}\gamma_{\alpha\beta} \Delta \right) \left(\bar{\Phi}_{(e)} + \frac{1}{\kappa^2} F \right) + \mathcal{A}_{\alpha\beta} \Psi_{(e)}. \quad (6.12)$$

Taking the divergence and then the rotation of this equation, we obtain

$$\Delta \Delta \Psi_{(e)} = 0. \quad (6.13)$$

Since Δ is the Laplacian associated with the flat metric, we easily show that for any solution to (6.13), there exists a function \mathcal{I} such that

$$\mathcal{A}_{\alpha\beta} \Psi_{(e)} = \left(D_\alpha D_\beta - \frac{1}{2}\gamma_{\alpha\beta} \Delta \right) \mathcal{I}. \quad (6.14)$$

This implies that we may choose $\Psi_{(e)} = 0$ without loss of generality, $F_{\alpha\beta}$ is given by

$$F_{\alpha\beta} = \left(D_\alpha D_\beta - \frac{1}{2}\gamma_{\alpha\beta} \Delta \right) (\bar{\Phi}_{(e)} + \mathcal{I}), \quad (6.15)$$

and Eq. (6.12) is reduced to

$$\left(D_\alpha D_\beta - \frac{1}{2}\gamma_{\alpha\beta} \Delta \right) \left(\bar{\Phi}_{(e)} + \mathcal{I} + \frac{1}{\kappa^2} F \right) = 0. \quad (6.16)$$

The divergence of Eq. (6.16) is given by

$$D_\alpha \Delta \left(\bar{\Phi}_{(e)} + \mathcal{I} + \frac{1}{\kappa^2} F \right) = 0. \quad (6.17)$$

This can be easily integrated as

$$C = \Delta (\bar{\Phi}_{(e)} + \mathcal{I}) - F, \quad (6.18)$$

where C is a constant of integration and we have used Eq. (5.27). Substituting this into Eq. (6.16), we obtain

$$\left(D_\alpha D_\beta - \frac{1}{2}\gamma_{\alpha\beta} \Delta \right) (\Delta + \kappa^2) (\bar{\Phi}_{(e)} + \mathcal{I}) = 0. \quad (6.19)$$

On the other hand, substituting Eq. (6.18) into Eq. (5.27), we obtain

$$\Delta (\Delta + \kappa^2) (\bar{\Phi}_{(e)} + \mathcal{I}) = \kappa^2 C. \quad (6.20)$$

From Eqs. (6.19) and (6.20), we obtain

$$D_\alpha D_\beta (\Delta + \kappa^2) (\bar{\Phi}_{(e)} + \mathcal{I}) = \frac{\kappa^2 C}{2} \gamma_{\alpha\beta}. \quad (6.21)$$

Using a function \mathcal{G} that satisfies the equation

$$D_\alpha D_\beta \mathcal{G} = \frac{1}{2} \gamma_{\alpha\beta} C, \quad (6.22)$$

Eqs. (6.18) and (6.21) can be written as

$$D_\alpha D_\beta (\Delta + \kappa^2) (\bar{\Phi}_{(e)} + \mathcal{I} - \mathcal{G}) = 0, \quad (6.23)$$

$$F = \Delta (\bar{\Phi}_{(e)} + \mathcal{I} - \mathcal{G}). \quad (6.24)$$

Here, we note that the existence of a solution to Eq. (6.22) is confirmed by explicit calculation, introducing an explicit coordinate system on $(\mathcal{M}_2, \gamma_{ab})$. Further, Eq. (6.15) can be written

$$F_{\alpha\beta} = \left(D_\alpha D_\beta - \frac{1}{2} \gamma_{\alpha\beta} \Delta \right) (\bar{\Phi}_{(e)} + \mathcal{I} - \mathcal{G}) \quad (6.25)$$

using Eq. (6.22).

Integrating Eq. (6.23), we obtain

$$(\Delta + \kappa^2) (\bar{\Phi}_{(e)} + \mathcal{I} - \mathcal{G}) = E_\alpha X^\alpha, \quad (6.26)$$

where E_α represents the component of a constant vector, and the vector X^α satisfies $D_\alpha X^\beta = \delta_\alpha^\beta$. Then, defining the function $\Phi_{(e)}$ by

$$\Phi_{(e)} := \bar{\Phi}_{(e)} + \mathcal{I} - \mathcal{G} - \frac{1}{\kappa^2} E_\beta X^\beta, \quad (6.27)$$

Eqs. (5.27), (5.28) and (5.29) are reduced to the single equation

$$(\Delta + \kappa^2) \Phi_{(e)} = 0, \quad (6.28)$$

and the gauge invariant metric perturbation variables F and $F_{\alpha\beta}$ are given by

$$F = \Delta \Phi_{(e)}, \quad F_{\alpha\beta} = \left(D_\alpha D_\beta - \frac{1}{2} \gamma_{\alpha\beta} \Delta \right) \Phi_{(e)}, \quad (6.29)$$

respectively. This scalar variable $\Phi_{(e)}$ represents gravitational waves correspond to the even mode gravitational waves in Ref. 1). Thus, using the solution to Eq. (6.28), we obtain the $\kappa \neq 0$ mode solution of $O(\epsilon\lambda)$ perturbations.

Substituting Eq. (6.29) into Eqs. (5.29) and (5.31), we directly see that

$$V^a = 0 = \Sigma. \quad (6.30)$$

Thus, the string does not bend in the $\kappa \neq 0$ mode perturbations at $O(\epsilon\lambda)$. This conclusion is different from that in our previous works, but it can be inferred from them.¹⁾ Actually, the conclusion obtained here can also be obtained by taking the limit in which the background deficit angle vanishes. Hence, we find that $\kappa \neq 0$ mode gravitational waves do not couple to the string motion, at least at $O(\epsilon\lambda)$. These gravitational waves have nothing to do with the string oscillation in this

perturbative treatment. This conclusion also implies that at least at first order in the string oscillation amplitude, the string oscillations do not produce gravitational waves that propagate to regions far from the string.

Finally, we note that the solutions to Eqs. (5.27)–(5.33) describe the free propagation of gravitational waves. These gravitational wave solutions also exist at both $O(\epsilon)$ and $O(\lambda)$. At both $O(\epsilon)$ and $O(\lambda)$, these gravitational waves also have nothing to do with the oscillations of the string. For this reason, we have imposed the condition that such gravitational waves exist at neither $O(\epsilon)$ nor $O(\lambda)$.

6.2. $\kappa = 0$ mode solutions

For $\kappa = 0$ mode perturbations, the $O(\epsilon\lambda)$ Einstein equations are given by (5.55)–(5.61). These equations are also almost the same as the linearized Einstein equations for the $\kappa = 0$ modes in Ref. 2) as we noted. However, in this subsection, we show that Eqs. (5.55)–(5.61) include solutions describing the string oscillation without gravitational waves. These oscillatory solutions correspond to the pp-wave exact solutions on the spacetime with deficit angle, as in our previous paper. These oscillatory solutions also coincide with the oscillations of the test string. Hence, the absence of the dynamical degree of freedom of gravitating string oscillations is not contrary to the dynamics of the test string, at least in the infinite string case. Further, from the derivation of the solutions, we can easily see that there are no other oscillatory solutions that describes the oscillations of an infinite string. To show this, we only need to derive the solutions to Eqs. (5.55)–(5.61).

In this paper, we derive the solutions only on the support of V_α and Σ (i.e., the support of ρ) and in the vacuum region (i.e., outside of the support of ρ), respectively. These solutions correspond to the solutions describing the string oscillations with gravitational wave propagation in Ref. 2). Strictly speaking, to show the existence of a solution describing string oscillations without gravitational waves, we should construct global perturbative solutions on \mathcal{M} by matching these solutions at the surface of the support of ρ . As shown in Ref. 2), this can be accomplished using the Israel junction conditions.¹⁶⁾ On the other hand, the perturbed Einstein equations (5.55)–(5.61) and those in Ref. 2) are both apply to the case of an arbitrary distribution of ρ , and their are nearly identical. Further, Israel's conditions guarantee only that the matching is not contrary to the Einstein equations. Therefore, it is obvious that the matching conditions derived from the $O(\epsilon\lambda)$ Israel's junction conditions also have the same as those in Ref. 2). Hence, to clarify the string oscillations without gravitational waves and compare with the results in Ref. 2), it is enough to derive the solutions in the two regions, i.e., on the support of ρ and outside of this support. From these solutions, we can easily see that the string can oscillate without gravitational waves in the case of a flat spacetime background.

First, we consider the solutions to Eqs. (5.58) and (5.60). Since we impose the condition that the metric perturbation H and H_α are regular on the space \mathcal{M}_2 , we obtain the solution to these equations

$$H_\alpha = 0 = H. \quad (6.31)$$

Then, Eq. (5.56) becomes trivial. Since $H_{\alpha\beta}$ is traceless, as expressed by Eq. (5.59),

we decompose the matrix $H_{\alpha\beta}$ as

$$H_{\alpha\beta} = \left(D_\alpha D_\beta - \frac{1}{2} \gamma_{\alpha\beta} \Delta \right) \Phi_{(\kappa 0)} + \mathcal{A}_{\alpha\beta} \Psi_{(\kappa 0)}. \quad (6.32)$$

Substituting Eqs. (6.32) and (6.31) into Eq. (5.57), we obtain

$$V_\alpha = \frac{1}{2} D_\alpha \Delta \Phi_{(\kappa 0)} + \frac{1}{2} \mathcal{E}_{\beta\alpha} D^\beta \Delta \Psi_{(\kappa 0)}. \quad (6.33)$$

This expression of V_α shows that $\Phi_{(\kappa 0)}$ and $\Psi_{(\kappa 0)}$ correspond to the irrotational and rotational parts of the matter velocity, respectively. Further, using Eqs. (6.32) and (6.33), we easily see that Eq. (5.55) is trivial. Finally, Eq. (5.61) gives an expression of Σ in terms $\Phi_{(\kappa 0)}$:

$$\Sigma = -\Delta \Delta \Phi_{(\kappa 0)}. \quad (6.34)$$

Equations (6.32), (6.33), and (6.34) are solutions to Eqs. (5.58) and (5.60) and have forms similar to the corresponding equations in Ref. 2).

When the displacement V_α vanishes, i.e., outside of the support of ρ , (6.33) yields $\Delta \Delta \Psi_{(\kappa 0)} = 0$. This implies that we may choose $\Psi_{(\kappa 0)} = 0$ without loss of generality, as in the case of $\kappa \neq 0$ mode perturbations [see Eq. (6.14)]. Further, Eq. (6.33) also yields $\Delta \Phi_{(\kappa 0)} = 0$, i.e.,

$$\Phi_{(\kappa 0)} = A_0 \ln r + \sum_{m=1}^{\infty} (A_m r^{-m} + B_m r^m) e^{im\phi}, \quad (6.35)$$

where $r := \sqrt{x^2 + y^2}$ and $\phi := \arctan(y/x)$ are the usual radial and azimuthal coordinates on the two-dimensional flat space, respectively [see Eq. (3.2)]. Then, Eq. (6.32) reduces to

$$H_{\alpha\beta} = D_\alpha D_\beta \Phi_{(\kappa 0)}. \quad (6.36)$$

From the above, we see that in the derivation of the solutions describing by Eqs. (6.32)–(6.34), (6.35) and (6.36), there is no delicate problem due to the fact that we used a background spacetime that is different from those in Ref. 2). To study the global solutions for $\kappa = 0$ mode perturbations, we may consider the expressions of the solutions in Ref. 2) with a vanishing background deficit angle (or a vanishing background curvature). In particular, in the thin string situation, only the $m = 1$ mode in Eq. (6.35) (i.e., the dipole deformation) is relevant to the string displacement. Unlike in the curved background case, the solution (6.36) for the $m = 1$ mode vanishes in this case, due to the absence of the background curvature. On the other hand, the displacement of the thin string defined in Ref. 2) gives a finite contribution and it does not depend on the thickness of the string, unlike in the curved background case. This implies that we can define the displacement of the thin string at any point on the support of the $O(\lambda)$ string energy density ρ , although in previous works we defined it on the boundary surface of the support of ρ . Then, we can derive the solutions describing the string oscillations without gravitational waves in the flat spacetime background. These solutions merely represent the oscillations of a test string. Hence, we can conclude that the absence of a dynamical degree of freedom

of gravitating string oscillations is not contradict to the dynamics of a test string, at least in the infinite string case.

We note that $\Phi_{(\kappa 0)}$ in (6.36) corresponds to the gravitational waves that propagate along the string. Further, we can easily see that this solution is just an exact solution to the Einstein equation that is called a “cosmic string traveling waves.”¹⁷⁾ The cosmic string traveling wave is the pp-wave exact solution. The test string oscillations obtained above represent just one limit of this pp-wave solution.

Finally, the solution (6.36) with (6.35) in the case $V^\alpha = 0 = \Sigma$ describes the propagation of free gravitational waves along \mathcal{M}_1 . When $V^\alpha = 0 = \Sigma$ at any point on \mathcal{M}_2 , we impose the regularity condition at $r = 0$ and $r \rightarrow \infty$ on the gauge invariant metric perturbation $H_{\alpha\beta}$. With this condition, only the $m = 2$ mode gives a constant contribution to $H_{\alpha\beta}$. This is just plane wave propagation of gravitational wave along \mathcal{M}_1 . These gravitational wave solutions also exist at $O(\epsilon)$ and $O(\lambda)$. However, these gravitational waves have nothing to do with the oscillations of the string. We thus see that at $O(\epsilon)$ and $O(\lambda)$, we have imposed the condition such that such gravitational waves do not exist.

§7. Summary and discussion

In this article, we have considered the perturbative oscillations of an infinite Nambu-Goto string. We developed the two parameter gauge invariant perturbation technique on a flat spacetime background. This perturbation theory includes two infinitesimal perturbation parameters, the string oscillation amplitude ϵ and the string energy density λ . We have considered the Einstein equations of $O(\epsilon)$, $O(\lambda)$, and $O(\epsilon\lambda)$.

In spite of the difference of the background spacetime, the Einstein equations of $O(\epsilon\lambda)$ are almost the same as the linearized Einstein equations in our previous papers.^{1), 2)} From these equations, we can obtain the solutions those describe the string oscillations without gravitational waves. From this fact, we conclude that the $m = 1$ mode cosmic string traveling wave should be regarded as the string oscillation, as discussed by Vachaspati and Garfinkle.¹⁷⁾ This solution coincides with that in our previous works²⁾ in the artificial limit in which the background deficit angle vanishes. Further, we have seen that there is no other oscillatory solutions that describes the oscillations of an infinite string. Since the existence of the pp-wave solution is closely related to the specific symmetry of the spacetime, it is also natural to conjecture that such solutions do not exist in more generic situations.

We should emphasize that the existence of the solution describing the string oscillations without gravitational waves is due to the fact that we have chosen a Minkowski spacetime as a background for the perturbation. This solution corresponds to that describing the thin string oscillation with the propagation of gravitational waves along the string in Ref. 2). Further, the results obtained here are less accurate than those obtained in Ref. 2), because the results in Ref. 2) can only be obtained from the infinite sum of $O(\epsilon\lambda^n)$ terms in the perturbative treatment developed here. Therefore, the conclusion in Ref. 2) is more accurate picture than that based on the dynamics of a test string, i.e., we conclude that *the dynamical*

degree of freedom of string oscillations is that of gravitational waves, and an infinite string can oscillate, but these oscillations simply represent the propagation of the gravitational waves along the string, which corresponds to cosmic string traveling waves (the exact solution). We have seen that these conclusions do not contradict to the results based on analyses developed here.

Further, we point out that the equation of motion of $O(\epsilon)$ is given by

$$\frac{\partial}{\partial \epsilon} K^a = -\mathcal{D}^b \mathcal{D}_b \hat{V}^a = 0. \quad (7.1)$$

This is identically the equation of motion for a test string to first order in the string oscillation amplitude. The solutions to this equation are consistent with the solutions to the Einstein equation of $O(\epsilon\lambda)$. Hence, the test string dynamics of $O(\epsilon)$ are consistent with the $O(\epsilon\lambda)$ Einstein equations. Further, the oscillations of the string, which are the solutions to Eq. (7.1) do not produce gravitational waves that propagate to regions far from the string, at least at $O(\epsilon\lambda)$.

We also note that this result is consistent with the power of the gravitational wave emitted from an infinite Nambu-Goto string derived by Sakellariadou,¹⁸⁾ which is based on the energy momentum tensor of an oscillating test string. She derived the power of the gravitational waves emitted by the helicoidal standing oscillations of an infinite string. In her result, the second order term of the string oscillation amplitude does not appear in the power of the gravitational waves. The second order [$O(\epsilon^2)$] term in the power of gravitational waves corresponds to the first order [$O(\epsilon)$] of the string oscillation amplitude. From her analyses, it is not clear if the absence of an $O(\epsilon^2)$ term in the power of gravitational waves is due to the helicoidal standing oscillations of the string, which is a special solution to the equation of motion of the Nambu-Goto string. However, our result shows that the absence of an $O(\epsilon^2)$ term in the power of gravitational waves is generic for gravitational waves emitted from an oscillating infinite string. Our result implies that string oscillation of $O(\epsilon)$ does not produce gravitational waves that reach regions far from the string, even if the string oscillations are very complicated.

We should stress that the consistency of the dynamics of a gravitating string and that of a test string discussed here is due to the existence of the cosmic string traveling wave solution. We note that the model studied here includes traveling waves, and the string can continue to oscillate, due to the existence of the pp-wave exact solution (cosmic string traveling wave). We also note that the existence of traveling waves has not been confirmed in the other exactly soluble models of gravitating Nambu-Goto walls.^{6),8)} In this sense, the model considered here might be an exceptional case. If there do not exist traveling waves in more generic situations of gravitating Nambu-Goto membranes, there is no guarantee that the oscillatory behavior of gravitating Nambu-Goto membranes is approximated by that of a test membrane. Further, we have found that there is no dynamical degree of freedom of the string oscillations, except for the cosmic string traveling waves. Hence, we can conclude that if the existence of traveling waves is not confirmed, the oscillatory behavior of gravitating Nambu-Goto membranes becomes highly nontrivial, and there is no guarantee that the estimate of the power of the gravitational waves emitted

from extended objects based on the test membrane dynamics is valid. Of course, it may result that these two treatments are consistent as seen here. Therefore, it would be interesting to consider the other models in which the perturbation theory on Minkowski spacetime is well-defined and for which the existence of traveling wave solutions along the membrane has not yet been confirmed. We leave this topic for future investigations.

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Appendix A

—— A More Precise Description of “Thick String” ——

Here, we give a more precise description of the “thick string” and its intrinsic metric q_{ab} studied in the main text. Since the world sheet of the usual Nambu-Goto string is a two-dimensional timelike surface embedded in the four-dimensional spacetime, we first consider the decomposition of the spacetime and its tangent space into a lower dimensional surface. Next, we consider the energy-momentum tensor for a “thick” Nambu-Goto membrane and its equation of motion derived from the divergence of this tensor. In this paper, the thickness of membranes is introduced in such a manner that the equation of motion of membranes is maintained. Through this introduction of the thickness of the membrane, we obtain the energy-momentum tensor for the “regularized Nambu-Goto membrane.” In this appendix, we consider the embedding of the M -dimensional surface Σ_M into the $N + M$ -dimensional spacetime (\mathcal{M}, g_{ab}) to guarantee the extension to any dimensional hypersurface embedded in a higher-dimensional spacetime.

A.1. Decomposition of the spacetime and metric

Let $x : \mathcal{M} \rightarrow \mathcal{R}^m$ be a coordinate system on \mathcal{M} . We introduce M functions σ^i ($i = 0, \dots, M - 1$), so that the hypersurface $x^\mu = x^\mu(\sigma^i)$ ($\mu = 0, \dots, N + M - 1$) is Σ_M . We also consider N functions ξ^α ($\alpha = M, \dots, M + N - 1$) such that all ξ^α are constant on Σ_M and all the one-forms $(d\xi^\alpha)_a := \nabla_a \xi^\alpha$ are linearly independent of each other. Further, choosing the functions σ^i ($i = 0, \dots, M - 1$) so that the one-forms $(d\sigma^i)_a := \nabla_a \sigma^i$ and $(d\xi^\alpha)_a$ are independent of each other. Then, the set of functions $\{\sigma^i, \xi^\alpha\}$ is a coordinate system on \mathcal{M} (at least in the neighborhood of Σ_M). Here, we consider the tangent space spanned by $(d\xi^\alpha)_a$ and introduce the orthonormal basis $(\theta^\alpha)_a$ of this tangent space by constructed from linear combinations of $(d\xi^\alpha)_a$; i.e., $g^{ab}(\theta^\alpha)_a(\theta^\beta)_b = \delta^{\alpha\beta}$, where $\delta^{\alpha\beta}$ is the N -dimensional Kronecker delta.

The metric induced on Σ_M from the metric g_{ab} on \mathcal{M} is defined by

$$q_{ab} := g_{ab} - \delta_{\alpha\beta}(\theta^\alpha)_a(\theta^\beta)_b. \quad (\text{A.1})$$

We also define the tensor

$$\gamma_{ab} := \delta_{\alpha\beta}(\theta^\alpha)_a(\theta^\beta)_b = g_{ab} - q_{ab}. \quad (\text{A}\cdot 2)$$

We note that both $q_a{}^b := q_{ac}g^{cb}$ and $\gamma_a{}^b := \gamma_{ac}g^{cb}$ have the properties of the projection operators:

$$q_a{}^b q_b{}^c = q_a{}^c, \quad \gamma_a{}^b \gamma_b{}^c = \gamma_a{}^c, \quad \gamma_a{}^b q_b{}^c = 0. \quad (\text{A}\cdot 3)$$

The operator $q_a{}^b$ projects the vectors in \mathcal{M} into Σ_M , and the operator $\gamma_a{}^b$ projects to the complement space of Σ_M . In the coordinate system $\{\sigma^i, \xi^\alpha\}$, the spacetime metric g_{ab} and its inverse g^{ab} are decomposed as

$$g_{ab} = \gamma_{\alpha\beta}(d\xi^\alpha)_a(d\xi^\beta)_b + q_{ij}((d\sigma^i) + \beta_\gamma^i(d\xi^\gamma))_a((d\sigma^j) + \beta_\delta^j(d\xi^\delta))_b, \quad (\text{A}\cdot 4)$$

$$g^{ab} = \gamma^{\alpha\beta} \left(\frac{\partial}{\partial \xi^\alpha} - \beta_\alpha^i \frac{\partial}{\partial \sigma^i} \right)^a \left(\frac{\partial}{\partial \xi^\beta} - \beta_\beta^j \frac{\partial}{\partial \sigma^j} \right)^b + q^{ij} \left(\frac{\partial}{\partial \sigma^i} \right)^a \left(\frac{\partial}{\partial \sigma^j} \right)^b, \quad (\text{A}\cdot 5)$$

where

$$q_{ij} := q_{ab} \left(\frac{\partial}{\partial \sigma^i} \right)^a \left(\frac{\partial}{\partial \sigma^j} \right)^b, \quad \gamma^{\alpha\beta} := g^{ab}(d\xi^\alpha)_a(d\xi^\beta)_b. \quad (\text{A}\cdot 6)$$

The matrices q^{ij} and $\gamma_{\alpha\beta}$ are the inverses of q_{ij} and $\gamma^{\alpha\beta}$, respectively. The coefficient β_α^i corresponds to the shift vector in the ADM decomposition.

For an arbitrary vector field t^a that satisfies $t^a = q_b{}^a t^b$, the covariant derivative \mathcal{D}_a associated with the metric q_{ab} is defined by

$$\mathcal{D}_a t^b := q_a{}^c q_d{}^b \nabla_c t^d. \quad (\text{A}\cdot 7)$$

Similarly, for an arbitrary vector field n^a that satisfies $n^a = \gamma_b{}^a n^b$, the covariant derivative D_a associated with the metric γ_{ab} is defined by

$$D_a n^b := \gamma_a{}^c \gamma_d{}^b \nabla_c n^d. \quad (\text{A}\cdot 8)$$

We also note that these derivatives satisfy $\mathcal{D}_a q_{bc} = 0$ and $D_a \gamma_{bc} = 0$, respectively. These definitions of the covariant derivatives are naturally extended to tensor fields of arbitrary rank.

Here, we introduce the second fundamental form by

$$K_a{}^b{}_c := q_a{}^e q_d{}^b \nabla_e q_c{}^d. \quad (\text{A}\cdot 9)$$

Using this second fundamental form, the relation between the covariant derivative \mathcal{D}_a and ∇_a is given by

$$X^b \nabla_b Y^a = X^b \mathcal{D}_b Y^a + X^c Y^d K_{cd}{}^a \quad (\text{A}\cdot 10)$$

for any vector field $X^a = q_b{}^a X^b$ and $Y^a = q_b{}^a Y^b$, where the first term in the right-hand side of Eq. (A·10) is the component of $X^b \nabla_b Y^a$ tangential to the tangent space of Σ_M , and the second term, $X^c Y^d K_{cd}{}^a$, is the normal component.

It is instructive to consider the trace of this extrinsic curvature in the coordinate system in which the spacetime metric g_{ab} is given by

$$g_{ab} = \left\{ \gamma_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu} + q_{ij} \left(\frac{\partial \sigma^i}{\partial x^\mu} + \beta_\gamma^i \frac{\partial \xi^\gamma}{\partial x^\mu} \right) \left(\frac{\partial \sigma^j}{\partial x^\nu} + \beta_\delta^j \frac{\partial \xi^\delta}{\partial x^\nu} \right) \right\} (dx^\mu)_a (dx^\nu)_b. \quad (\text{A}\cdot 11)$$

In this coordinate system, the extrinsic curvature is given by

$$K_{ab}{}^c =: K_{\mu\nu}{}^\rho (dx^\mu)_a (dx^\nu)_b \left(\frac{\partial}{\partial x^\rho} \right)^c, \quad (\text{A}\cdot 12)$$

and its trace is given by

$$K^c := g^{ab} K_{ab}{}^c = q^{ab} K_{ab}{}^c =: K^\mu \left(\frac{\partial}{\partial x^\mu} \right), \quad (\text{A}\cdot 13)$$

where K^μ is given by

$$K^\mu = \frac{1}{\sqrt{-q}} \frac{\partial}{\partial \sigma^i} \left(\sqrt{-q} q^{ij} \frac{\partial x^\mu}{\partial \sigma^j} \right) + \Gamma_{\kappa\sigma}^\mu q^{ij} \frac{\partial x^\kappa}{\partial \sigma^i} \frac{\partial x^\sigma}{\partial \sigma^j}. \quad (\text{A}\cdot 14)$$

A.2. Infinitesimally thin Nambu-Goto membrane

Here, we consider the action of an infinitesimally thin Nambu-Goto membrane. The Nambu-Goto action is given by the area of the world sheet of the membrane:

$$S = -\mu \int d^M \sigma \sqrt{-q}, \quad q = \det(q_{ij}). \quad (\text{A}\cdot 15)$$

Here, μ is the energy density of the membrane. The variation of this action with respect to $x^\mu(\sigma^i)$ gives the equation of motion of the membrane:

$$K^a = 0. \quad (\text{A}\cdot 16)$$

The energy-momentum tensor of the membrane can be found by varying the action (A.15) with respect to the metric g_{ab} . We have

$$T^{ab} \sqrt{-g} = -2 \frac{\delta S}{\delta g_{ab}} = \mu \int d^M \sigma \sqrt{-q} q^{ab} \delta^{M+N}(x^\alpha - x^\alpha(\sigma^i)), \quad (\text{A}\cdot 17)$$

where $\delta^{M+N}(x^\alpha - x^\alpha(\sigma^i))$ is the Dirac delta function defined on the entire spacetime (\mathcal{M}, g_{ab}) by

$$\int_{\mathcal{M}} d^{N+M} x \delta^{M+N}(x^\alpha - x^\alpha(\sigma^i)) = 1. \quad (\text{A}\cdot 18)$$

Choosing the coordinate system $\{\xi^\alpha, \sigma^i\}$, we obtain

$$T^{ab} \sqrt{-g} = \mu \delta^N(\xi^\alpha) \sqrt{-q} q^{ab} (\sigma_i, \xi^\alpha = 0). \quad (\text{A}\cdot 19)$$

Here, we note that the motion of the membrane is characterized by N functions ξ^α , because the world sheet of the membrane is the surface on which all ξ^α are constant. If we consider the perturbed world sheet, we need only consider the perturbations of the functions ξ^α and the spacetime metric g_{ab} , as in the main text. From these perturbations, we can directly calculate the perturbation of the intrinsic metric q_{ab} of the membrane world sheet.

A.3. Regularized Nambu-Goto membrane

On the basis of the above energy-momentum tensor, (A.19), for an infinitesimally thin Nambu-Goto membrane, we consider the energy-momentum tensor for a regularized Nambu-Goto membrane with which the dynamics of an infinitesimally thin Nambu-Goto membrane are unchanged. The energy-momentum tensor (A.19) has the following properties: (i) T^{ab} is proportional to the induced metric q^{ab} on the world sheet of the membrane; and (ii) the support of T^{ab} is confined to the world sheet (which is proportional to $\delta^N(\xi^\alpha)$). These properties should be separated when we consider a regularized membrane. We note that the first property is essential to the equation of motion derived from the divergence of the energy-momentum tensor. The second property simply reflects the fact that the membrane is infinitesimally thin. Because we will study the dynamics of Nambu-Goto membrane using a regularized membrane, we need only change property (ii) leaving property (i) as it is.

Now, we consider the energy-momentum tensor for the regularized membrane with which the dynamics of this membrane are the same as those of the Nambu-Goto membrane. This tensor is the following:

$$T^{ab} = -\rho q^{ab}. \quad (\text{A.20})$$

Here, ρ is a scalar function on \mathcal{M} whose support is the compact region \mathcal{D} in the complement space of Σ_M . The “thick string world sheet” is given by $\mathcal{D} \times \Sigma_M$. Physically, the function ρ represents the energy density of the Nambu-Goto membrane. The usual line energy density μ in (A.15) is given by

$$\mu = \int_{\mathcal{D}} d^N \xi \sqrt{\gamma} \rho. \quad (\text{A.21})$$

Similarly, according to the definition (A.1), q^{ab} is extended to the region $\mathcal{D} \times \Sigma_M$. The extended tensor q^{ab} becomes a tensor of rank M on $\mathcal{D} \times \Sigma_M$. The extrinsic curvature K_{abc} is also extended to a tensor on $\mathcal{D} \times \Sigma_M$ by substituting the extended q_{ab} into Eq. (A.9). These extended versions of q^{ab} and K_{abc} are regarded as the induced metric and the extrinsic curvature of each hypersurface $\xi^\alpha = \text{const.}$ at each point on \mathcal{D} , respectively. As shown in the main text (Eqs. (2.3) and (2.4)), the divergence of the energy momentum tensor (A.20) guarantees that the dynamics of the regularized membrane are the same as those of the Nambu-Goto membranes, though the additional continuity equation should be taken into account.

Finally, we comment on the meaning of the “thickness” of the regularized Nambu-Goto membrane. This “thickness” is determined by the length that characterizes the support \mathcal{D} of the energy density ρ . For example, the region \mathcal{D} may be chosen as the closure of the N -dimensional open ball of radius r_* in an appropriate chart. The natural choice of the measure of the radius r_* is the circumferential radius of this open ball. With this choice, the original world sheet of the infinitesimally thin Nambu-Goto membrane is obtained by choosing $r_* = 0$ and the “thickness” of membrane is characterized by the radius r_* . We emphasize that we consider the situation in which the thickness r_* is much smaller than the curvature scale of the bending mem-

brane in order to obtain the dynamics of a thin membrane and also to avoid Israel's paradox. This approach was applied to an infinite string in our previous works.^{1),2)}

Appendix B

— Perturbative Ricci curvatures —

Here, we list the perturbative curvatures at each order of the perturbative treatment with Minkowski background spacetime. As seen in §5, the metric perturbations at each order are decomposed into gauge invariant parts and gauge variant parts as (3·11), (3·12), and (3·19). These decompositions are useful to calculate the perturbative curvature at each order. Further, the perturbative curvature at each order is represented by the tensors $\mathcal{H}_{ab}{}^c$, $\mathcal{L}_{ab}{}^c$ and $\mathcal{K}_{ab}{}^c$ defined by (4·2), (4·5) and (5·1), respectively, and gauge variant part ${}^{(\epsilon)}X_a$ and ${}^{(\lambda)}X_a$ of the metric perturbation.

The perturbative curvatures of each order are given by

$$\left. \frac{\partial}{\partial \epsilon} R_{abc}{}^d \right|_{\epsilon=\lambda=0} = -2\nabla_{[a} \mathcal{H}_{b]c}{}^d, \quad (\text{B} \cdot 1)$$

$$\left. \frac{\partial}{\partial \lambda} R_{abc}{}^d \right|_{\epsilon=\lambda=0} = -2\nabla_{[a} \mathcal{L}_{b]c}{}^d, \quad (\text{B} \cdot 2)$$

$$\left. \frac{\partial}{\partial \epsilon} R_{ab} \right|_{\epsilon=\lambda=0} = -2\nabla_{[a} \mathcal{H}_{c]b}{}^c, \quad (\text{B} \cdot 3)$$

$$\left. \frac{\partial}{\partial \lambda} R_{ab} \right|_{\epsilon=\lambda=0} = -2\nabla_{[a} \mathcal{L}_{c]b}{}^c, \quad (\text{B} \cdot 4)$$

$$\begin{aligned} \left. \frac{\partial^2}{\partial \lambda \partial \epsilon} R_{ab} \right|_{\epsilon=\lambda=0} &= -2\nabla_{[a} \mathcal{K}_{c]b}{}^c + 2\mathcal{H}_{[b}{}^{cd} \mathcal{L}_{c]ad} + 2\mathcal{L}_{[b}{}^{cd} \mathcal{H}_{c]ad} \\ &\quad + \mathcal{L}_{(\epsilon)X} \left(\left. \frac{\partial}{\partial \lambda} R_{ab} \right|_{\epsilon=\lambda=0} \right) - \mathcal{H}^{cd} \left(\left. \frac{\partial}{\partial \lambda} R_{acbd} \right|_{\epsilon=\lambda=0} \right) \\ &\quad + \mathcal{L}_{(\lambda)X} \left(\left. \frac{\partial}{\partial \epsilon} R_{ab} \right|_{\epsilon=\lambda=0} \right) - \mathcal{L}^{cd} \left(\left. \frac{\partial}{\partial \epsilon} R_{acbd} \right|_{\epsilon=\lambda=0} \right). \end{aligned} \quad (\text{B} \cdot 5)$$

We can confirm these formulae in the straightforward way. Using these formulae, we evaluated the Einstein equations order by order in the main text.

Appendix C

— Harmonics for the mode expansion —

To carry out the evaluation of the perturbation at each order, it is useful to consider the mode expansion of the perturbative variables. In this appendix, we introduce the harmonics for the expansion. The harmonics introduced here are useful to analyze the perturbative oscillations of an infinite string, as done in the main text.

In the main text, we assumed that a string without oscillations is straight and that the intrinsic metric on the string world sheet is given by

$$q_{ab} = -(dt)_a(dt)_b + (dz)_a(dz)_b. \quad (\text{C} \cdot 1)$$

This metric is also that on \mathcal{M}_1 in the main text. We consider the mode expansion associated with the symmetries of this metric q_{ab} . The natural scalar harmonic function on the string world sheet is given by

$$S := e^{-i\omega t + ik_z z}. \quad (\text{C}\cdot 2)$$

From this scalar harmonic function, we can define the vector and tensor harmonics.

To consider the vector and tensor harmonics, it is necessary to distinguish the perturbative modes according to whether or not κ , defined by

$$\kappa^2 := \omega^2 - k_z^2, \quad (\text{C}\cdot 3)$$

vanishes. The $\kappa = 0$ modes correspond to the perturbative modes propagates along the string, and the $\kappa \neq 0$ modes correspond to the other dynamical modes. We should also comment that a different treatment is necessary to consider cylindrical and stationary perturbations, as seen in §5.

C.1. $\kappa \neq 0$ mode harmonics

First, we introduce the $\kappa \neq 0$ mode harmonics. Here, we denote all perturbative variables formally by Q . Let us consider the perturbative variables Q which satisfy the equations

$$\mathcal{D}_c Q \neq 0, \quad \mathcal{D}^c \mathcal{D}_c Q \neq 0. \quad (\text{C}\cdot 4)$$

The first condition in Eqs. (C·4) implies that Q is not constant and the second condition implies that the variable Q is expanded by the eigen functions with the non-vanishing eigen value κ [see Eq.(C·3)] of the derivative operator $\mathcal{D}^c \mathcal{D}_c$. According to the conditions (C·4), the $\kappa \neq 0$ mode perturbative variables are defined. If the perturbative variable Q is a scalar, vector or tensors of second rank on \mathcal{M}_1 , and if it satisfies Eqs. (C·4), it can be expanded in the harmonics

$$S := e^{-i\omega t + ik_z z}, \quad (\text{C}\cdot 5)$$

$$V_{(o1)}^a := \epsilon^{ab} \mathcal{D}_b S, \quad (\text{C}\cdot 6)$$

$$V_{(e1)}^a := q^{ab} \mathcal{D}_b S, \quad (\text{C}\cdot 7)$$

$$T_{(e0)ab} := \frac{1}{2} q_{ab} S, \quad (\text{C}\cdot 8)$$

$$T_{(e2)ab} := \left(\mathcal{D}_a \mathcal{D}_b - \frac{1}{2} q_{ab} \mathcal{D}^c \mathcal{D}_c \right) S, \quad (\text{C}\cdot 9)$$

$$T_{(o2)ab} := -\epsilon_{c(a} \mathcal{D}_{b)} \mathcal{D}^c S, \quad (\text{C}\cdot 10)$$

where $\epsilon^{ab} = q^{ac} q^{bd} \epsilon_{cd}$ and $\epsilon_{ab} = (dt)_a (dz)_b - (dz)_b (dt)_a$ are two-dimensional totally antisymmetric tensors.

C.2. $\kappa = 0$ mode harmonics

Next, we introduce the $\kappa = 0$ mode harmonics. Let us consider the perturbative variables, which are denoted formally by Q , satisfy the conditions

$$\mathcal{D}_c Q \neq 0, \quad \mathcal{D}^a \mathcal{D}_a Q = 0. \quad (\text{C}\cdot 11)$$

The first condition in Eqs. (C·11) implies that Q is not constant and the second condition implies that the variable Q is expanded by the eigen functions with the vanishing eigen value κ [see Eq.(C·3)] of the derivative operator the derivative operator $\mathcal{D}^c\mathcal{D}_c$. According to the conditions (C·11), the $\kappa = 0$ mode perturbative variables are defined. The scalar harmonic function for the $\kappa = 0$ mode perturbations is given by Eq. (C·5) with the condition $\kappa = 0$. To introduce vector and tensor harmonics of $\kappa = 0$ mode, we first introduce null vectors $k^a(=q_b{}^ak^b)$ and $l^a(=q_b{}^al^b)$ defined by

$$k_a := -i\mathcal{D}_a S, \quad k^a k_a = 0, \quad l_a k^a := -2\omega^2, \quad l^a l_a = 0. \quad (\text{C}\cdot 12)$$

Using these null vectors, we introduce vector and tensor harmonics as follows:

$$S := e^{-i\omega(t+\nu z)}, \quad (\text{C}\cdot 13)$$

$$V_{(e1)}^a := q^{ab}\mathcal{D}_b S, \quad (\text{C}\cdot 14)$$

$$V_{(l1)}^a := il^a S, \quad (\text{C}\cdot 15)$$

$$T_{(e0)ab} := \frac{1}{2}q_{ab}S, \quad (\text{C}\cdot 16)$$

$$T_{(e2)ab} := \mathcal{D}_a\mathcal{D}_b S, \quad (\text{C}\cdot 17)$$

$$T_{(l2)ab} := -l_a l_b S, \quad (\text{C}\cdot 18)$$

where $\nu = \pm 1$. If the perturbative variable Q is a scalar, vector or tensor of second rank on \mathcal{M}_1 , and if Q satisfies the conditions (C·11), it can be expanded in the harmonics (C·13)–(C·18).

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